Nature's Measuring Tape: A Cognitive Basis for Adaptive Utility

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Abstract

This paper provides a new approach to a utility function, by deriving it from deeper first principles. Using a minimum set of cognitive tools, one can evaluate choice options and their expectations using solely information from one's past and present environments. The adaptive evaluation procedure results in one's utility function being isomorphic to a perceived rank of its magnitude within a reference set, and thus provides a mechanism for one's apparent attitudes towards risk to be shaped entirely by one's experiences, memory, and cognitive imperfections. This conceptually simple, but powerful, framework provides support for neoclassical utility theory as a special case, but also rationalizes reference-dependence and social comparisons. Cognitive imperfections have unexpected effects on choice under uncertainty and economic welfare. The proposed parsimonious model links recent developments in economics, psychology, and neuroscience, and shows that environmental context, memory, and cognitive processes may interact in non-trivial and economically relevant ways.

Keywords: Biological basis of behavior, reference-dependent preferences, random utility, procedural approach, memory limitations; perceptual imperfections, Weber-Fechner law, range-frequency theory, decision-by-sampling, overconfidence, well-being.

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1 Introduction

How big is an apple? How high is a salary? How intense is an experience? How young is a person? How frequent is a name? How long is a piece of string? How can one tell?

Consider Robinson Crusoe on a desert island. One day, a coconut falls off a tree. How happy would be Crusoe with that coconut? Undoubtedly, that would depend on the size of the coconut. And whether Crusoe would be content with the coconut in his hands rather than reaching for another coconut on the tree, would also depend - at least in part - on the sizes of the two coconuts. Yet he has no measuring tape, no scales, no other modern measuring devices. Similarly, our prehistoric ancestors, as well as all living organisms, have been making such economically relevant decisions, and thus must have been able to evaluate the magnitudes of their choice options, - all in the absence of the modern measurement technology. So, how would Nature enable individuals to make magnitude evaluations, and what is the relationship of such magnitude evaluations to a utility function?

The concept of a utility function is fundamental to economics and behavioral sciences, yet the process of utility acquisition is still not well understood. The present paper undertakes a novel approach to a concept of a utility function. Simply put, before one can construct a utility function $U(x)$, one needs to evaluate, or *measure*, the size of x. I thus explore the process and tools by which a given magnitude x can be evaluated. Following Robson [2001, 2002], I assume that, rather than endowing living organisms with all necessary information, Nature provides the living organisms with "tools" that enable one to extract information from one's environment and experience. In order for such tools to be adapted to deal with more evolutionarily recent tasks, the candidate cognitive tools have to be independent of the nature of the measurement.

I identify two well-documented domain-independent cognitive tools and, using a parsimonious mathematical model, show that one can evaluate a magnitude of an item entirely using *ordinal comparisons* which, by means of a *frequency* (proportion) tool, are keyed into a universal cardinal scale. By calculating how frequently a given object "wins" a pairwise ordinal "tournament" against all other objects in the reference set, one can map an arbitrary set of modalities (such as quantities, sizes, weights, durations, luminosities, and so on) onto the interval $[0, 1]$. Such process of evaluation by ordinal rank was advanced informally by psychologists Stewart, Chater and Brown [2006].

The resultant adaptive evaluation, which is the expected outcome of pairwise comparisons, is a non-decreasing function of a magnitude, and thus rationalizes "relatively more is better" preferences defined on a reference set. Once the reference set changes, so does the adaptive utility. Specifically, a given magnitude is evaluated higher if the reference set is positively rather than negatively skewed, leading to a possibility of non-reflexivity, non-transitivity, and preference reversal (as Tversky and Kahneman [1991] pointed out, such "preference anomalies" are a common consequence of a change in a "reference state"). The model of adaptive evaluation thus provides a new approach to the concept of utility function, by deriving it from deeper first principles, and providing the mechanism which allows one to construct reference-dependent utility, and to explain how one's apparent attitudes towards risk could be shaped entirely by one's experiences, memory, and cognitive imperfections.

Using this conceptually simple, but powerful, framework, I show that a (neoclassical) context-independent utility function may arise as a *special case* of adaptive evaluation procedure when one evaluates a magnitude relatively to a remembered sample and one has a long memory. In contrast, if one's memory is bounded, context effects may arise. One thus would expect the context effects to be less pronounced when an agent faces familiar objects (e.g. orange juice) rather than when one faces relatively unfamiliar objects (e.g. caviar).

Consistently with the principle of efficient use of information, the reference sample can be used two ways - both to evaluate magnitudes and, similarly to the case-based decision theory of Gilboa and Schmeidler [1995, 2003], to estimate probabilities in a frequentist way. As every context can be seen as a gamble which involves a random draw of a single element, one can derive an adaptive equivalent of expected utility, or *context evaluation*, which is an expected evaluation of a random element drawn from a given context.

I show *the fundamental property of adaptive evaluation*, namely that if an agent relies solely on the information from a single environment, and has a perfect frequency tool, he is indifferent among the contexts (or the gambles the contexts provide) if and only if the imperfections of his ordinal tool satisfy certain conditions. That is, once the conditions on the ordinal tool are satisfied, an individual would be equally happy with any gamble as long as he can only keep a single reference set in his memory at a time, but remembers his evaluations of each entire context. Yet despite this indifference across contexts, his choice between a contextual gamble and its expected magnitude with certainty will be different across contexts. Specifically, an agent may exhibit risk aversion (i.e. choose the expected magnitude rather than a gamble) whenever the expected magnitude exceeds the median (as typically happens in positively skewed distributions), and vice versa.

The exposure to other environments tends to have a spill-over effect on one's expected evaluation. Thus, one's evaluation of a random environment will decrease (or increase) whenever, in addition, one is exposed to an alternative environment which is stochastically better (worse), and, just as the expected utility theory predicts, a first order stochastically higher gamble would be chosen. In general, under the assumptions of perfect cognitive tools and memory, the cases when the theory of adaptive magnitude evaluation and the expected utility theory generate different predictions tend to involve empirically uncommon contextual distributions.

It is documented that the two cognitive tools employed here are subject to cognitive limitations and thus may lead to informational inaccuracies. To accommodate such perceptual errors for a continuous distribution case, I employ a convolution/scale mixture technique whereby an adaptive evaluation can be expressed as either a sum or a product of two independent random variables, namely the reference magnitude and the outcome of pairwise comparison tournament. The adaptive magnitude evaluation with perfect cognitive tools provides a useful benchmark, as it is equal to the "veridical" (true) rank in the distribution of reference magnitudes, and thus is *optimal* in the sense of Robson [2001], minimizing mistakes in binary choices.

The convolution model allows one to explore of the effects of imperfections in cognitive tools on magnitude evaluation. An imperfect magnitude evaluation is only partially affected by the veridical rank, thus obscuring the empirical relationship between the environment and one's evaluation function. One particularly notable result states that when the reference context is uniform on $(0, 1)$, an *arbitrary* form of multiplicatively imperfect ordinal tool would result in adaptive evaluation exhibiting non-increasing marginal evaluation. Such cognitive distortion of the mental line would lead to apparent risk aversion in an environment where risk neutrality would have been optimal in a sense of Robson [2001]. The continuous case model also highlights that the intuitively plausible results tend to be associated with ordinal comparison tools which exhibit upward comparison inaccuracies (rather than "fair" judgments or downward inaccuracies).

Despite the asocial nature of cognitive processes, the procedure allows for built-in social comparisons. Since all economically relevant magnitudes (such as income, consumption, etc.) are attributable to individuals who possess them, the relevant magnitude evaluations necessarily involve interpersonal comparisons. Yet cognitive imperfections may prevent individuals from assessing their possessions accurately. Specifically, when individuals make more inaccurate ordinal comparisons when comparing their possessions to those "above" them, than to those "below" them, their evaluations of own possessions are higher than "true" evaluation, leading to greater satisfaction with their possessions. In the presence of such "upward inaccuracy" bias, equality and economic growth may be associated with greater welfare. Yet the result may reverse if the bias exhibits instead "downward inaccuracy". In contrast, if the ordinal tool is "fair" (or symmetric), economic processes such as economic growth and income redistribution may surprisingly be welfare neutral. That implies that when adaptive evaluation is optimal in a sense of Robson [2001], social welfare would be independent of economic growth and redistribution. Furthermore, as judgment of own skill relatively to the others involves interpersonal comparisons, the proposed evaluation model allows one to relate individual difficulties in making upward ordinal comparisons with the observed patterns of overconfidence in relative skill judgment.

2 Adaptive Magnitude Evaluation: Discrete Case

Consider Robinson Crusoe on a desert island. There is a single coconut tree on the island, and Crusoe can see coconuts in a variety of sizes on that tree. One day, a coconut falls off the tree. How could Crusoe evaluate the size of this coconut? Clearly, if the coconut is the smallest one among the ones he can see, it is the least valuable. If instead it is the largest one, it is the most valuable. If it is neither smallest nor largest, its evaluation is in-between.

This paper argues that only two primitive tools are sufficient to "measure" a magnitude against a reference set. One such primitive tool - ordinal comparisons - enables one to tell whether a magnitude is larger, smaller, or equal to another magnitude. Brannon [2002] found that 9-11 months old human infants are able to discriminate between small arrays of objects, while Feigenson, Carey and Spelke [2002] suggested that human infants develop abilities to discriminate continuous variables (such as areas, sizes, densities) even earlier, while Xu [2003] reported that large numerosities are discriminated differently from the smaller ones. Walsh [2003] further suggested that time, space and numbers involve similar neural mechanisms.

The other primitive tool is frequency (proportion) estimation. Hasher and Zacks [1979, 1984] suggested that humans update frequencies-of-occurrence automatically and accurately, while Hintzman [1988] proposed a model of retrospective updating of frequency-of-occurrence. Since human children and some animals are equipped with a mental system for counting (Gallistel and Gelman [1992]), natural frequencies can be stored in a numerical format (Jonides and Jones [1992]). Gigerenzer and Hoffrage [1995] and Cosmides and Tooby [1996] further argued that natural frequency format is computationally simple, and thus is evolutionary advantageous way to calculate probabilities (proportions).

I thus hypothesize that Crusoe can construct an evaluation function using only two cognitive tools: first, by conducting *pairwise comparison* "tournaments" between the *target* coconut and each of the coconuts which he can see on the tree, and second, by evaluating the target magnitude by the *frequency*, or *proportion* of pairwise comparison tournaments the target coconut "wins". As the adaptive evaluation procedure is context-dependent, Crusoe's evaluation is based on the size distribution of coconuts on his island, and will change if he were on another island. For example, Crusoe would be less happier with a 3 inch coconut on an island A where a typical coconut is 5 inch radius rather than on another island B where a typical coconut is 2 inch radius - simply because coconuts smaller than 3 inch are more frequent on island B. Thus it is possible to find a banana such that Crusoe will be happy to trade the 3-inch coconut for this banana on island A, but on island B he would be happy to trade this banana back for the 3-inch coconut - an apparent preference reversal.

2.1 Adaptive Evaluation Algorithm

Consider an *environment* S consisting of magnitudes of (potentially) observable objects (e.g. sizes of physical goods such as coconuts, houses, cars, incomes, and so on, or magnitudes of a characteristic such as height, beauty, intelligence, etc.), with veridical (true) distribution $F : S \to [0, 1]$. An individual observes a *context* S_N, which is a sample of N observations, independently and identically drawn from set S with distribution F. This context S_N constitutes the individual's *reference set*.

The individual is endowed with two cognitive tools. The *ordinal comparison* tool $\precsim$ allows him to compare any two magnitudes x and y in his reference set S_N, whether y is perceived to be "smaller" or "bigger" than x - i.e. either $y \prec x$ or $y \succ x$. Whenever the individual perceives that neither $y \prec x$ nor $y \succ x$, x and y are perceived to be *similar*, or $x \sim y$.[1] In that case, y is judged to be "smaller" with probability $p \in [0, 1]$, and "larger" with probability $1 - p$. The ordinal comparison tool may be imperfect in a sense that two veridically (truly) distinct magnitudes may appear to be indistinguishable. However, a magnitude evaluation with perfect ordinal tool will serve as a useful benchmark throughout the paper.

Definition 1 *The ordinal comparison tool $\precsim$ is perfect if only veridically equal magnitudes are perceived to be similar, i.e. $y \sim x \Leftrightarrow y = x$ for any S_N.*

Definition 2 *The ordinal comparison tool $\precsim$ involves fair (symmetric) judgment if, for every x, a similar magnitude has an equal chance to be judged larger or smaller, or $p = \frac{1}{2}$.*

The *frequency tool* # allows the individual to estimate natural frequencies (counts) of items with particular attributes, and estimate relative frequencies (proportions) of those attributes within a (sub)population. The frequency (proportion) may be imperfect as an individual may make mistakes in frequency counts and proportion estimations. However, for most part of the paper, it will be assumed that the frequency (proportion) tool is perfect.

Definition 3 *The frequency (proportion) tool # is perfect if estimates of natural and relative frequencies-of-occurrence are accurate, i.e. for two sets of objects with particular attributes $S_M \subseteq S_N$ with cardinalities $M \leq N$, the natural frequency estimates are $\#^P(y \in S_M) = M$ and $\#^P(y \in S_N) = N$, and the relative frequency estimate is $\frac{M}{N}$.*

Using these two cognitive tools, the individual evaluates an arbitrary target magnitude $x \in S_N$ relatively to the reference set S_N using the following algorithm.

Adaptive Evaluation Algorithm: *Suppose an individual faces a reference set (or context) S_N and is endowed with ordinal comparison tool $\precsim$ and perfect frequency (proportion) tool $\#^P$. Then the evaluation of the target magnitude x can be constructed as follows:*[2]

Step 1 *Using the ordinal comparison tool $\precsim$, compare the "target" magnitude x to every magnitude y in the reference set S_N, and divide the reference set S_N into two subsets: set $\hat{S}_{\prec}(x) = S_{y \prec x} \cup S^{\prec}_{y \sim x}$ (containing magnitudes which are perceived to be smaller*

[1] The ordinal comparison tool $\precsim$ can be thought as a perceptual semiorder in a sense of Luce [1956].

[2] One can modify the above proportion-based algorithm into a natural frequencies-based version, with the magnitude evaluation being "normalized" by the cardinality of the reference set $\#_{y \in S_N}$.

and those which are are perceived to be similar but judged to be smaller) and set $\hat{S}_{\succ}(x) = S_{y \succ x} \cup S_{y \sim x}^{\succ}$ *(containing magnitudes which are perceived to be larger and those which are are perceived to be similar but judged to be larger).*[3]

Step 2 *Using the perfect frequency (proportion) tool* $\#^P$*, evaluate the target magnitude* x *relatively to the reference set* S_N *by estimating how often every* $y \in S_N$ *is judged to be smaller, i.e.*

$$\hat{I}_{S_N}(x) = \frac{N_{y \prec x} + N_{y \sim x}^{\prec}}{N_{y \in S_N}} \tag{1}$$

Following the above algorithm, the individual evaluates an arbitrary magnitude x relatively to the reference set S_N by the perceived proportion of reference magnitudes which he judges to be smaller (or no larger) than x, or by its *perceived rank* in a *perceived reference distribution* $\hat{F}_{S_N}(x) = Prob(y \hat{\preceq} x)$, or a probability that magnitude x is judged to be less than an arbitrary other reference magnitude y.[4]

The set of judged-to-be-smaller magnitudes $S_{\hat{\prec}}(x) = S_{y \prec x} \cup S_{y \sim x}^{\prec}$ includes a subset $S_{y \sim x}^{\prec}$ containing reference magnitudes y which are similar to x but are randomly judged to be smaller with probability $p \in [0, 1]$. Thus, for each magnitude x, its *realized evaluation* $\hat{I}_{S_N}(x)$ is generated by a binomial process with parameters $N_{y \sim x}$ and p. If the individual evaluates magnitude x many times using the above algorithm with perfect frequency tool, on average, the proportion p of similar reference magnitudes would be judged to be smaller. Let the *adaptive evaluation* of magnitude x be the expected outcome of pairwise comparison, or the expected realized evaluation of this magnitude:

$$I_{S_N}(x) = E\hat{I}_{S_N}(x) = \frac{N_{y \prec x}}{N} + p\frac{N_{y \sim x}^{\prec}}{N} \tag{2}$$

The realized evaluation $\hat{I}_{S_N}$ can thus be decomposed into its expectation, given by the adaptive evaluation I_{S_N}, and a random noise variable ϵ:

$$\hat{I}_{S_N} = I_{S_N} + \epsilon$$

Following the random utility models pioneered by Thurstone [1927], the rest of the paper will concentrate on the adaptive evaluation I_{S_N} as a candidate for a utility function.

2.2 Magnitude Evaluation with Perfect Cognitive Tools

The case when both cognitive tools are perfect provides a useful benchmark as well as clarifies the relationship of the present model to the previous literature. When both tools are perfect,

[3] Most results would not change if the target magnitude x were excluded from the reference set S_N, except those involving stochastically ordered contexts require equally-sized reference sets.

[4] Note that, by construction, the perceived distribution $\hat{F}$ does not allow for "ties" so that $\hat{F}_{S_N}(x) = Prob(y \hat{\preceq} x)$ equals to $\hat{F}_{S_N}^{-}(x) = Prob(y \hat{\prec} x)$.

the realized evaluation can be written as

$$\hat{I}^P_{S_N}(x) = \frac{N_{y<x} + \hat{N}^<_{y=x}}{N}$$

so the adaptive evaluation (which is expectation of realized evaluation) can be written as

$$I^P_{S_N}(x) = E\hat{I}^P{}_{S_N}(x) = \frac{N_{y<x}}{N} + p\frac{N_{y=x}}{N} = pF_N(x) + (1-p)F^-_N(x) \tag{3}$$

where $F_N(x) = Prob(x \leq y)$ is the *empirical distribution* of magnitudes in context S_N, and $F^-_N(x) = Prob(x < y)$, for $x, y \in S_N$. Thus, when both tools are perfect, the *perceived rank* (and thus adaptive evaluation) of magnitude x in reference set S_N is given by $\hat{F}_{S_N}(x) = pF_N(x) + (1-p)F^-_N(x)$.

Example 1 *Consider a reference set $A = \{10, 11, 12, 12, 18, 18, 18, 18, 19, 20\}$ with $N_A = 10$, and suppose both cognitive tools are perfect, Then, for any $p \in [0, 1]$, the realized evaluations and adaptive evaluations of each magnitude are:*

$\hat{I}^P{}_A(10) \in \left\{\frac{0}{10}; \frac{1}{10}\right\} \Rightarrow I^P_A(10) = \frac{p}{10}$ $\quad$ $\hat{I}^P{}_A(18) \in \left\{\frac{4}{10}; \frac{5}{10}; \frac{6}{10}; \frac{7}{10}; \frac{8}{10}\right\} \Rightarrow I^P_A(18) = \frac{4+4p}{10}$

$\hat{I}^P{}_A(11) \in \left\{\frac{1}{10}; \frac{2}{10}\right\} \Rightarrow I^P_A(11) = \frac{1+p}{10}$ $\quad$ $\hat{I}^P{}_A(19) \in \left\{\frac{8}{10}; \frac{9}{10}\right\} \Rightarrow I^P_A(19) = \frac{8+p}{10}$

$\hat{I}^P{}_A(12) \in \left\{\frac{2}{10}; \frac{3}{10}; \frac{4}{10}\right\} \Rightarrow I^P_A(12) = \frac{2+2p}{10}$ $\quad$ $\hat{I}^P{}_A(20) \in \left\{\frac{9}{10}; \frac{10}{10}\right\} \Rightarrow I^P_A(20) = \frac{9+p}{10}$

Note that the adaptive evaluation function $I^P_A(x)$ is non-decreasing in magnitude x. The next result follows from Tversky and Kahneman [1991], who considered complete and transitive "relatively more is better" *preference structure* $\succeq_{S_N}$ with *reference state* S_N.

Proposition 1 *Complete and transitive "relatively more is better" preference structure $\succeq_{S_N}$ is rationalizable by the adaptive evaluation $I^P_{S_N}(x)$ as defined in (3).*

The following result specifies the conditions resulting in the adaptive evaluation to be a veridical (true) empirical rank $F_N(x)$, and to be optimal in a sense of Robson [2001].

Definition 4 *An evaluation function $I(x)^R$ is optimal in a sense of Robson [2001] if it is equal to the veridical (true) distribution $F(x)$ of magnitudes in the environment S, so that a chance of mistaken choice of a smaller magnitude in a binary choice is minimized.*

Proposition 2 *Suppose both cognitive tools are perfect. In addition, suppose that all reference magnitudes y which are similar to x are judged to be smaller with certainty, i.e. $p = 1$ for all x and y. Then*
(i) the realized evaluation $\hat{I}_{S_N}$ is equal to adaptive evaluation $I_{S_N}(x)$, and both are isomorphic

to the empirical distribution function $F_N(x)$;
(ii) as the reference set S_N becomes large, it approaches in the limit to the environment S, so that the adaptive evaluation $I_{S_N}(x)$ converges uniformly to the veridical (true) distribution $F(x)$ and thus to the Robson-optimal evaluation $I_S^R(x)$.

Proof: By definition, the ordinal rank of x in S_N is given by the *empirical distribution function* $F_N(x) = \frac{N_{y \leq x}}{N} = \frac{N_{y<x}+N_{y=x}}{N}$, or, $F_N(x)$ is the frequency of magnitudes in S_N that do not exceed x. By assumption of perfect tools and $p = 1$ for all x and y, $\hat{N}^+_{y \sim x} = N_{y=x}$, so that $I_{S_N}(x) = \hat{I}_{S_N}(x) = \frac{N_{y<x}+N_{y=x}}{N} = \frac{N_{y \leq x}}{N} = F_N(x)$. As all observations in S_N are i.i.d. draws from $F(x)$, the rest follows from Glivenko-Cantelli Theorem (e.g. Durrett [1996]). As Robson [2001] and Netzer [2009] showed, a mistake of choosing a smaller magnitude from a set of two draws from an environment $F(x)$ is minimized when the evaluation (utility) function equals to $F(x)$. ∎

Thus, if all similar magnitudes are judged to be smaller, then the adaptive evaluation I_{S_N} is isomorphic to an empirical distribution, or cumulative density function. The isomorphism of a utility function and a cumulative density function was first noticed by Van Praag [1968], and further explored by Kapteyn [1985 and references therein]. Gilboa and Schmeidler [2003] extended the case-based decision theory to represent "at least as likely" binary relation with ranking alternatives by their empirical frequencies. Stewart, Chater and Brown [2006] suggested informally that one can evaluate attribute values (such as money amounts, time, or probability) by their ordinal rank in a distribution of a sample from memory.

2.3 Adaptive Evaluation with Ordinal Imperfections

Among the advantages of the adaptive evaluation model is a potential to clarify the role of cognitive imperfections on adaptive magnitude evaluation. Specifically, cognitive imperfections distort one's mental line and lead to magnitude evaluation to be suboptimal in a sense of Robson [2001].

As psychologists discovered in the XIX century, subjects tend not to notice the difference between two magnitudes if this difference is less than a certain threshold, and such minimum noticeable difference is proportional to the stimulus level. This observation, widely known as Weber-Fechner psychophysical law of just noticeable differences (e.g. Link [1992]) is the psychologist's counterpart to the law of diminishing marginal utility. For example, most people would notice the difference between having one dollar bill and two dollar bills in one's wallet, but would hardly notice the difference between having 67 and 68 dollar bills. In fact, Dehaene, Dupoux and Mehler [1990] found that when comparing two-digit Arabic numerals, such as 64 and 65, subjects made more than 10% mistakes, but such mistakes drop dramatically once the numerals are far apart. This phenomenon was recently found on the brain level, as Nieder and Miller [2003] found that the representation of numerosities is "compressed" in a monkey brain, prompting Dehaene [2003] to advance an argument in support of a logarithmic mental number line to represent the Weber-Fechner law.[5] Such findings open up a possibility that the ordinal comparison tool may be imperfect in the sense that any two magnitudes which belong to an interval of *doubt* are *perceived* to be similar.

Consider a veridical distribution F on context S with N elements, and suppose that F is discrete, with J "bins" $\mathcal{B}_i, i = 1, \ldots, J$ ranked in the ascending order of their veridical (true) magnitudes, i.e. for any $x \in \mathcal{B}_i$ and $y \in \mathcal{B}_j$, $x \lesseqqgtr y \Leftrightarrow i \lesseqqgtr j$. Denote the number of elements in bin $\mathcal{B}_i$ as N_i, and the total number of elements which belong to the smallest h bins as $\Sigma_h = \sum_{i=1,\ldots,h} N_i$, so that context S consisting of J bins contains $\Sigma_J = \sum_{i=1,\ldots,J} N_i$ elements in total. Thus, $F(h) = Prob(x \leq x^h) = \frac{\Sigma_h}{\Sigma_J}$ is the value of cumulative magnitude distribution in context S corresponding to magnitude in bin h, and $x^h = \max x \in \mathcal{B}_1 \cup \ldots \cup \mathcal{B}_h$.

Assume a perfect frequency tool. Let the individual's ability to tell apart the magnitudes x and y to depend only on the veridical (true) magnitudes x and y, and not on the context S. That is, for some i, there exist $k \geq 0$ such that such that for all $x \in \mathcal{B}_i$ and all $y \in \mathcal{B}_{i-j}$ with $j = \{0, \cdots, k\}$, the elements of $k+1$ bins are indistinguishable, or $x \sim y$. If $k=0$, the ordinal tool is perfect. Otherwise, with $k > 0$, an imperfect ordinal tool results in the perceived distribution $\hat{F}$ to be a *coarsening* of the veridical distribution F, by "fusing" $k+1$ veridical bins $\mathcal{B}_j, j = i-k, \ldots, i$ into a perceived bin $\hat{\mathcal{B}}_{i-k:i}$ with $\hat{\Sigma}_{i-k:i} = \sum_{j=i-k,\ldots,i} N_j$ elements. Thus, an adaptive evaluation of each element x belonging to the "fused" bin $\hat{\mathcal{B}}_{i-k:i}$ is

$$I(i-k) = \cdots = I(i) = \frac{\Sigma_{i-k-1} + p\hat{\Sigma}_{i-k:i}}{\Sigma_J} = pF(i) + (1-p)F(i-k-1) \tag{4}$$

(Note, if both cognitive tools are perfect, $k = 0$, so that $I^P(i) = pF(i) + (1-p)F(i-1)$.) If the extremes of the veridical magnitude range appear to have similar magnitudes, lowest magnitudes will tend to be overvalued and highest magnitude will tend to be undervalued.

Proposition 3 *Suppose the frequency tool is perfect. Consider context S with J magnitude bins described by veridical distribution $F(x)$. If an imperfect ordinal tool results in "fusing" the lowest $k_l > 1$ bins, then the lowest magnitude x_{min} is overvalued relatively to the Robson-optimal evaluation $F(x_{min})$, and if ordinal tool results in "fusing" the highest $k_h > 0$ bins, then the highest magnitude x_{max} is undervalued relatively to $F(x_{max})$.*

Proof: Since x_{min} belongs to the lowest bin $\mathcal{B}_1$, its adaptive evaluation is:

$$I(1) \quad = \quad pF(k_l) + (1-p)F(1) = F(1) + p(F(k_l) - F(1)) > F(1)$$

[5] Dehaene, Izard, Spelke, and Pica [2008] found that education has a "linearizing" effect on the mental number scale, as in Amazonian indigenous cultures the mental number line is found to be logarithmic, while in the Western societies it was found to be closer to the linear.

Since x_{max} belongs to the highest bin $\mathcal{B}_J$, its adaptive evaluation is:

$$I(J) \quad = \quad pF(J) + (1-p)F(J-k_h) = F(J) - (1-p)(F(J) - F(J-k_h)) < F(J) \ \blacksquare$$

Furthermore, as the example below shows, the stochastic nature of ordinal imperfections may result in smaller magnitudes having higher realized evaluations than larger magnitudes, with a possibility of apparent preference anomalies even for the same reference set.

Example 2 *Suppose that frequency tool is perfect. Consider an imperfect ordinal comparison tool resulting in any two magnitudes* x *and* y *perceived to be similar whenever the "veridical" magnitude difference does not exceed* 2*, (i.e.* $\mid x - y \mid \leq 2$*). Thus, in the context* A *from the Example 1, the individual perceives* $10 \sim 11 \sim 12$ *and* $18 \sim 19 \sim 20$*, with context* A *perceived to be* $A' = \{11, 11, 11, 11, 19, 19, 19, 19, 19, 19\}$*, resulting in*

$$\hat{I}_{A'}(x) \in \left\{ \frac{0}{10}; \frac{1}{10}; \frac{2}{10}; \frac{3}{10}; \frac{4}{10} \right\} \quad \text{for} \quad x \in \{10, 11, 12\}$$
$$\hat{I}_{A'}(x) \in \left\{ \frac{4}{10}; \frac{5}{10}; \frac{6}{10}; \frac{7}{10}; \frac{8}{10}; \frac{9}{10}; \frac{10}{10} \right\} \quad \text{for} \quad x \in \{18, 19, 20\}$$

Thus it is possible that the realized evaluation $\hat{I}$ *can be non-monotone in magnitude, for example,* $\hat{I}_{A'}(12) < \hat{I}_{A'}(11) < \hat{I}_{A'}(10) < \hat{I}_{A'}(100) < \hat{I}_{A'}(99) < \hat{I}_{A'}(98)$*. However, the adaptive evaluations non-decreasing in magnitude:*

$$I_{A'}(10) = I_{A'}(11) = I_{A'}(12) \quad = \quad \frac{4p}{10}$$
$$I_{A'}(18) = I_{A'}(19) = I_{A'}(20) \quad = \quad \frac{4+6p}{10}$$

*Here, relatively to the "true" evaluation of Example 1, ordinal imperfections lead to overvaluing "small" magnitudes (*10 *and* 11*) and undervaluing "large" ones (*19 *and* 20*).*

2.4 Adaptive Evaluation across Different Contexts

Would Robinson Crusoe's evaluations of coconuts change if there existed more than one coconut tree on his desert island? That depends on how much Crusoe remembers about the other trees on the island. It thus would be natural to distinguish Crusoe's "local" reference set which he would form if he were unaware or amnesiac, and his "global" reference set if all coconuts on all trees were part of his reference set. One would thus expect that his evaluation relatively to such "global" reference set would typically be different from evaluation relatively to just one "local" tree.

As Tversky and Kahneman [1991] pointed out, a "reference shift", or change in a reference state, often results in apparent preference anomalies. Thus, when the environment S changes, so does the reference set S_N, with the adaptive evaluation $I^P_{S_N}(x)$, exhibiting non-reflexivity, non-transitivity and preference reversals when the choices are compared across different reference states.

Example 3 *Suppose the frequency tool is perfect. Consider a set* $B = \{10, 11, 12, 12, 12, 18, 18, 18, 19, 20\}$ *with* $N_B = 10$. *If the ordinal tool is perfect, the realized magnitude evaluations are:*

$$\begin{array}{llll}
\hat{I^P}_B(10) \in \{\frac{0}{10}; \frac{1}{10}\} & \Rightarrow I^P_B(10) = \frac{p}{10} & \hat{I^P}_B(18) \in \{\frac{5}{10}; \frac{6}{10}; \frac{7}{10}; \frac{8}{10}\} & \Rightarrow I^P_B(18) = \frac{5+3p}{10} \\
\hat{I^P}_B(11) \in \{\frac{1}{10}; \frac{2}{10}\} & \Rightarrow I^P_B(11) = \frac{1+p}{10} & \hat{I^P}_B(19) \in \{\frac{8}{10}; \frac{9}{10}\} & \Rightarrow I^P_B(19) = \frac{8+p}{10} \\
\hat{I^P}_B(12) \in \{\frac{2}{10}; \frac{3}{10}; \frac{4}{10}; \frac{5}{10}\} & \Rightarrow I^P_B(12) = \frac{2+3p}{10} & \hat{I^P}_B(20) \in \{\frac{9}{10}; \frac{10}{10}\} & \Rightarrow I^P_B(20) = \frac{9+p}{10}
\end{array}$$

Suppose instead the ordinal tool is imperfect as in the Example 2, resulting in the context B to be perceived as $B' = \{11, 11, 11, 11, 11, 19, 19, 19, 19, 19\}$ *and thus*

$$\hat{I}_{B'}(x) \in \left\{\frac{0}{10}; \frac{1}{10}; \frac{2}{10}; \frac{3}{10}; \frac{4}{10}; \frac{5}{10}\right\} \quad \textit{for} \quad x \in \{10, 11, 12\}$$
$$\hat{I}_{B'}(x) \in \left\{\frac{5}{10}; \frac{6}{10}; \frac{7}{10}; \frac{8}{10}; \frac{9}{10}; \frac{10}{10}\right\} \quad \textit{for} \quad x \in \{18, 19, 20\}$$

so that the adaptive evaluations are:

$$I_{B'}(10) = I_{B'}(11) = I_{B'}(12) = \frac{5p}{10}$$
$$I_{B'}(18) = I_{B'}(19) = I_{B'}(20) = \frac{5+5p}{10}$$

Note that set B is stochastically smaller than set A in the Example 1, and, similarly, the perceived set B′ is stochastically smaller than perceived set A′ in the Example 2. Since large magnitudes are more frequent in set A than in set B, all elements of stochastically smaller set B are evaluated no lower than in set A - regardless of the properties of the ordinal tool.

Thus, the same absolute magnitude may have different adaptive evaluations in different contexts. Moreover, the adaptive evaluation exhibits well-documented context effect, namely, if one reference context stochastically dominates another (as in the Example 1), then all magnitudes in the stochastically lower context have adaptive evaluations which are no lower.

Proposition 4 *Suppose the frequency tool is perfect. If* $A \succeq_{FOSD} B$, *then* $I_A(x) \leq I_B(x)$.

Proof: Define the indiscriminability bins $\mathcal{B}_i$ on the union of the two contexts $A \cup B$ (with some bins being possibly empty in one context but not in both), and let J be the largest number of bins across both contexts. $S_A \succeq_{FOSD} S_B$ implies that $F_A(j) \leq F_B(j)$ for any $j = 1, \ldots, J$. Then, for every $x \in \mathcal{B}_i$, the difference in evaluations across the two contexts is

$$I_A(x) - I_B(x) = I_A(i) - I_B(i) = p(F_A(i) - F_B(i)) + (1-p)(F_A(i-k-1) - F_B(i-k-1)) \leq 0 \blacksquare$$

This systematic effect of a change in the reference set on subjects' evaluations has been documented in a number of psychological studies, including Parducci [1963, 1965], Stewart [2009], Olivola and Sagara [2009], Wood, Brown, and Maltby [2011], Ungemach, Stewart, and Reimers [2011], Stewart, Reimers, and Harris [2011].

Next, it would be of interest to compare one's "local" and "global" adaptive evaluations. Consider two sets A and B, containing N^A and N^B items, respectively. The adaptive evaluation $I_{A\cup B}(x)$ with perfect frequency tool relatively to a combined context $A \cup B$ is

$$I_{A\cup B}(x) = I_A(x)\frac{N^A}{N^A+N^B} + I_B(x)\frac{N^B}{N^A+N^B} \tag{5}$$

or one can write is as $I_{A\cup B}(i)$ to denote $I_{A\cup B}(x)$ for any $x \in \mathcal{B}_i$:

$$I_{A\cup B}(i) \quad = \quad \frac{I_A(i)N^A + I_B(i)N^B}{N^A+N^B}$$

Corollary 1 *Suppose the frequency tool is perfect. If* $A \succeq_{FOSD} B$*, then* $I_A(x) \leq I_{A\cup B}(x) \leq I_B(x)$*.*

In other words, one's adaptive evaluation of each magnitude is increased if stochastically dominated set is added to one's reference set. And vice versa, adding a stochastically dominant set to one's reference set decreases one's magnitude evaluations.

Example 4 *Suppose the frequency tool is perfect. Given the contexts A and B of Examples 1 and 3, construct* $A\cup B = \{10, 10, 11, 11, 12, 12, 12, 12, 12, 18, 18, 18, 18, 18, 18, 18, 19, 19, 20, 20\}$ *with* $N_{A\cup B} = 20$*. With perfect ordinal tool, the adaptive evaluations of each magnitude relatively to the "global" reference set are:*

$$\begin{array}{lll} I^P_{A\cup B}(10) = \frac{2p}{20} & I^P_{A\cup B}(11) = \frac{2+2p}{20} & I^P_{A\cup B}(12) = \frac{4+5p}{20} \\ I^P_{A\cup B}(18) = \frac{9+5p}{20} & I^P_{A\cup B}(19) = \frac{16+2p}{20} & I^P_{A\cup B}(20) = \frac{18+2p}{20} \end{array}$$

so that $A \succeq_{FOSD} B$ *results in* $I_A(x) \leq I_{A\cup B}(x) \leq I_B(x)$*. Observe that if the ordinal imperfection is as in the Example 2, the perceived "global" reference context* $A' \cup B'$ *will consist of 9 "small" items (with* $10 \sim 11 \sim 12$*) and 11 "large" items (with* $18 \sim 19 \sim 20$*), so that*

$$\begin{aligned} I_{A'\cup B'}(10) = I_{A'\cup B'}(11) = I_{A'\cup B'}(12) &= \frac{9p}{20} \\ I_{A'\cup B'}(18) = I_{A'\cup B'}(19) = I_{A'\cup B'}(20) &= \frac{9+11p}{20} \end{aligned}$$

Note that the nature of the ordinal tool does not affect the comparative statics, as elements belonging to stochastically smaller context are evaluated no lower.

Such behavior is well documented, e.g. in Parducci [1968], students judged poisoning a neighbour's dog to be a lesser crime then when poisoning the neighbour herself was on the list as well.

2.5 Context Evaluation

Suppose Robinson Crusoe who faces a single coconut tree which randomly sheds a single coconut per day. How could Crusoe evaluate such a lottery? Would Crusoe prefer to have a random coconut from that tree or the coconut with an average (veridical) magnitude? Note that the coconut tree serves as a database of coconut magnitudes, which Crusoe uses to evaluate the *size* of a given coconut. But at the same time, the coconut tree also provides Crusoe with a database for calculating frequencies of coconuts of various sizes, thus allowing to calculate a *probability* with which a coconut of a particular size might fall off the tree. Thus, Crusoe can use the Nature's tools to evaluate the entire tree as a gamble.

Furthermore, suppose there are two coconut trees at the opposite ends of the island, and Crusoe has to be next to the tree to pick up the fallen coconut before wild animals consume it. How would Crusoe decide which tree to choose? It turns out that when Crusoe faces a choice between two trees, in addition to his cognitive tools, his memory plays an important role. While Crusoe always remembers his evaluations of each tree, his choice will depend on whether he remembers the coconuts hanging on each tree whenever he walks between the two trees. With fair ordinal tool, an amnesiac Crusoe who only remembers coconuts on one tree at a time, will not care which tree to choose. In contrast, if Crusoe had a perfect memory about all coconuts on all trees, he would choose a stochastically bigger tree.

The case-based decision theory of Gilboa and Schmeider [1995, 2003] offers an alternative to expected utility theory where agents utilize a database of past experiences to make decisions. Rather than weighting utilities by probabilities, the weights are constructed based on the frequencies of appropriate experiences in the past. As I show here, such database stored in one's memory can be used efficiently as a source of information *both* for constructing an evaluation function and constructing a probability distribution.

Let us assume that the frequency tool is perfect, and allow for a possibility of ordinal imperfections. For any context S_N consisting of N items, for an element x belonging to the "fused" bin $\hat{\mathcal{B}}_{i-k:i}$ one can define the *perceived frequency* $\hat{f}_{S_N}(x)$ as

$$\hat{f}_{S_N}(x) = \frac{N_{y\sim x}}{N_{y\in S_N}} = \frac{\hat{\Sigma}_{i-k:i}}{\Sigma_J}$$

Given the adaptive evaluation $I_{S_N}(x)$ constructed on context S_N, one can write *context evaluation* $E_{S_N}(S_N)$ for context S_N as:

$$E_{S_N}(S_N) = \sum_{x\in S_N} I_{S_N}(x)\hat{f}_{S_N}(x)$$

Using the equation (4), one can write context evaluation $E_J = E_{S_J}(S_J)$ as:

$$E_J = \sum_{i=1,\ldots,J} I_i f_i = \sum_{i=1,\ldots,J} \frac{\Sigma_{i-k-1} + p\hat{\Sigma}_{i-k:i}}{\Sigma_J}\frac{\hat{\Sigma}_{i-k:i}}{\Sigma_J} \qquad (6)$$

or, using the alternative expression from (4), and taking into account that $F(J) = 1$, one can write the above as

$$\begin{aligned} E_J &= \sum_{i=1,..,J} [pF(i) + (1-p)F(i-k-1][F(i-k-1) - F(i)] = \\ &= 1 - p + (2vp-1) \sum_{i=1,..,J} [F(i)^2 - F(i)F(i-k-1)] \end{aligned} \tag{7}$$

Example 5 *Suppose both cognitive tools are perfect, and consider contexts A and B of Examples 1 and 4. Then, for any $p \in [0,1]$, the context evaluation of set A is:*

$$E_A(A) = \frac{p}{10}\frac{1}{10} + \frac{1+p}{10}\frac{1}{10} + \frac{2+2p}{10}\frac{2}{10} + \frac{4+4p}{10}\frac{4}{10} + \frac{8+p}{10}\frac{1}{10} + \frac{9+p}{10}\frac{1}{10} = \frac{38+24p}{100}$$

the context evaluation of set B is:

$$E_B(B) = \frac{p}{10}\frac{1}{10} + \frac{1+p}{10}\frac{1}{10} + \frac{2+3p}{10}\frac{3}{10} + \frac{5+3p}{10}\frac{3}{10} + \frac{8+p}{10}\frac{1}{10} + \frac{9+p}{10}\frac{1}{10} = \frac{39+22p}{100}$$

Note that these two distinct reference sets A and B have the same context evaluation equal to $\frac{1}{2}$ if the ordinal tool is fair, i.e. $p = \frac{1}{2}$.

The above example with fair ordinal tool demonstrates *the fundamental property of adaptive context evaluation*. If the frequency tool is perfect and if magnitude and frequency evaluations are based on the same context, then context evaluation is *context independent* if and only if judgment of similarly perceived magnitudes is fair.

Proposition 5 *Suppose the frequency tool is perfect. The context evaluation $E_S(S)$ is independent of the context S if and only if the probability p of judging a similar magnitude to be smaller is $p = \frac{1}{2}$. Moreover, $E_S(S) = \frac{1}{2}$ for any S iff $p = \frac{1}{2}$.*

Proof: From (7) one can observe that if $p = \frac{1}{2}$, then $E_{S_J}(S_J) = \frac{1}{2}$ for any context. Furthermore, if context evaluation $E_{S_J}(S_J)$ is context-independent, it must be that $p = \frac{1}{2}$, and thus $E_{S_J}(S_J) = \frac{1}{2}$. ∎

This is a striking result which only requires that the ordinal judgment is fair, and does not depend on other properties of the ordinal comparison tool. What is says that if an individual is not aware or does not remember about the existence of other contexts, and he has fair ordinal and perfect frequency tools, he will be indifferent among gambles as long each gamble is evaluated entirely relatively to itself. What is also important, if one faces a single stochastic context, one's context evaluation of that gamble is equivalent to the adaptive evaluation of the median magnitude.

However, individual's choice between a gamble and the average (veridical) magnitude will depend on the stochastic properties of the context. When distribution is positively skewed (so that larger magnitudes are less frequent), its median magnitude is smaller than mean magnitude, and the adaptive evaluation of such context is lower than the evaluation of the average magnitude. This can be seen as a form of risk aversion, as the expected value of a gamble is less than the value of its expectation. In contrast, if the distribution is negatively skewed (so that the larger magnitudes are more frequent), the expected value of the gamble is more than the value of its expectations, corresponding to risk loving. We thus have the following straightforward result.

Corollary 2 *If frequency tool is perfect, and consider a context S with magnitude distribution F_S, average magnitude $\mu = E(x)$ and median $m = F_S^{-1}(0.5)$. If the cognitive tools are perfect, then $I_S E(x) > (<) E I_S(S)$ iff $\mu > (<) m$.*

That is, one's choice between a gamble and its expectation could potentially be governed solely by the properties of the stochastic environment. However, the memory of previously encountered environments could be important for behavior in risky situations. To see this, suppose a individual faces a choice between two contexts A and B, with F_A and F_B. If his memory can only hold information about one context at a time, and if his ordinal tool is fair, then Proposition 5 would apply, as it holds for any context S. Instead, suppose the individual's memory is perfect, so that his database is a combined "global" set $A \cup B$. Moreover, suppose the individual's memory allows him to keep track of which element in his memory belongs to which context.[6] For the purposes of magnitude evaluation, the entire "global" reference set $A \cup B$ is relevant, and one can use equation (5). But, for the purposes of evaluation of the gambles that each context provides, only the subset of elements associated with a particular context $K = A, B$ is relevant as only those allow one to evaluate the probability of an element with a particular magnitude:

$$E_{A \cup B}(K) = \sum_{x \in K} I_{A \cup B}(x) \hat{f}_K(x) = \sum_{x \in K} \frac{I_A(x) N^A + I_B(x) N^B}{N^A + N^B} \frac{N^K_{y \sim x}}{N^K}$$

or, using the context-specific frequencies $f_K(i) = \frac{N_i^K}{N^K}$ for any $x \in \mathcal{B}_i$:

$$E_{A \cup B}(K) = \sum_{i=1,\dots,J} \frac{I_A(i) N^A + I_B(i) N^B}{N^A + N^B} \frac{N_i^K}{N^K}$$

In other words, the perceived distribution in context B affects the expected magnitude evaluation of context A, and vice versa. One thus would expect the choice between two contexts

[6] As Mullett and Tunney [2013] find, activations in ventro-medial prefrontal cortex (vmPFC) and the anterior cingulate cortex (ACC) are consistent with encoding "global" ordinal rank, while activations in ventral striatum and thalamus record "local" ordinal rank.

to be affected by the relationship between the two contexts, as the following proposition suggests. As the next result demonstrates, when stochastically lower values are added for evaluation purposes, their magnitude valuations which would have been higher in isolation provide a boost to the original magnitudes.

Proposition 6 *Suppose the frequency tool is perfect. If $A \succeq_{FOSD} B$, then $E_{A\cup B}(A) > E_A(A)$, $E_{A\cup B}(B) < E_B(B)$, and $E_{A\cup B}(A) > E_{A\cup B}(B)$.*

Proof: Since $A \succeq_{FOSD} B$ implies that $I_A(i) \leq I_B(i)$, contexts A and B are evaluated as

$$\begin{aligned} E_{A\cup B}(A) &= \sum_{i=1,\ldots,J} \frac{I_A(i)N^A + I_B(i)N^B}{N^A + N^B} \frac{N_i^A}{N^A} > \sum_{i=1,\ldots,J} I_A(i) \frac{N_i^A}{N^A} = E_A(A) \\ E_{A\cup B}(B) &= \sum_{i=1,\ldots,J} \frac{I_A(i)N^A + I_B(i)N^B}{N^A + N^B} \frac{N_i^B}{N^B} < \sum_{i=1,\ldots,J} I_B(i) \frac{N_i^B}{N^B} = E_B(B) \ \blacksquare \end{aligned}$$

That is, if bigger sizes are more common in one context than in the other (in a sense of first order stochastic dominance), the expected evaluation of stochastically bigger (smaller) context will be higher (lower) than what they would have been worth to the amnesiac individual. In other words, exposing one to information about an alternative context makes one better off if the alternative context is stochastically worse, and vice versa. Furthermore, as one would expect, the stochastically bigger context is more desirable.

Example 6 *Suppose the frequency tool is perfect, and consider sets A, B and $A \cup B$ of Examples 1, 3 and 4. Then, for perfect ordinal tool with any $p \in [0, 1]$, when magnitude evaluation is done relatively to the global set $A \cup B$, the context evaluation of the entire set $A \cup B$ is:*

$$E_{A\cup B}(A \cup B) = \frac{2p}{20}\frac{2}{20} + \frac{2+2p}{20}\frac{2}{20} + \frac{4+5p}{20}\frac{5}{20} + \frac{9+7p}{20}\frac{7}{20} + \frac{16+2p}{20}\frac{2}{20} + \frac{18+2p}{20}\frac{2}{20} = \frac{155+90p}{400}$$

while the context evaluation of set A when magnitude is evaluated based on the "global" reference set $A \cup B$ is:

$$E_{A\cup B}(A) = \frac{2p}{20}\frac{1}{10} + \frac{2+2p}{20}\frac{1}{10} + \frac{4+5p}{20}\frac{2}{10} + \frac{9+7p}{20}\frac{4}{10} + \frac{16+2p}{20}\frac{1}{10} + \frac{18+2p}{20}\frac{1}{10} = \frac{80+46p}{200}$$

and the context evaluation of set B is:

$$E_{A\cup B}(B) = \frac{2p}{20}\frac{1}{10} + \frac{2+2p}{20}\frac{1}{10} + \frac{4+5p}{20}\frac{3}{10} + \frac{9+7p}{20}\frac{3}{10} + \frac{16+2p}{20}\frac{1}{10} + \frac{18+2p}{20}\frac{1}{10} = \frac{75+44p}{200}$$

Clearly, for any $p \in [0, 1]$, $E_{A\cup B}(A) > E_A(A)$ and $E_{A\cup B}(B) < E_B(B)$. Furthermore, consider the imperfect ordinal tool as in the Example 2. Because the individual cannot distinguish *between* $10, 11$ *and* 12*, each of these veridical magnitudes are perceived to have identical frequencies* $\frac{4}{10}$ *in set* A'*. Similarly, each of* $18, 19$ *and* 20 *are perceived to appear with frequency* $\frac{6}{10}$*. Thus, context* A' *is evaluated as*

$$E_{A'}(A') = \frac{4p}{10} \cdot \frac{4}{10} + \frac{4+6p}{10} \cdot \frac{6}{10} = \frac{24+52p}{100}$$

and context B' *is evaluated as*

$$E_{B'}(B') = \frac{5p}{10} \cdot \frac{5}{10} + \frac{5+5p}{10} \cdot \frac{5}{10} = \frac{25+50p}{100}$$

The fundamental property holds, as for $p = \frac{1}{2}$*,* $E_{A'}(A') = E_{B'}(B') = \frac{1}{2}$*. However, relatively to the "global" comparison set* $A' \cup B'$*, the two "local" contexts have different evaluations:*

$$\begin{aligned} E_{A' \cup B'}(A') &= \frac{9p}{20}\frac{4}{10} + \frac{9+11p}{20}\frac{6}{10} = \frac{54+102p}{200} \\ E_{A' \cup B'}(B') &= \frac{9p}{20}\frac{5}{10} + \frac{9+11p}{20}\frac{5}{10} = \frac{45+100p}{200} \end{aligned}$$

Clearly, as $A' \succeq_{FOSD} B'$*, we have that* $E_{A' \cup B'}(A') > E_{A' \cup B'}(B')$ *for any* $p \in [0, 1]$*.*

In other words, the adaptive procedure results in intuitively plausible rankings of contexts, and such ranking does not depend on the nature of the ordinal imperfections.

2.6 Interaction Between Reference Context and Memory

Obviously, Robinson Crusoe may evaluate a target coconut not only relatively to the set of coconuts which are currently observable to him, but he can also base his evaluation on his memory of coconut sizes seen previously, or even imagined. How his memory may enter into his evaluation?

Both Gilboa and Schmeider [1995, 2003] and Stewart, Chater and Brown [2006] assume that an individual uses a database stored in one's memory. It thus may be only partially affected by the current environment, but it can also be affected by the past observations - and thus might be misremembered. Brown and Matthews [2011] extend the decision-by-sampling model to allow for memory to interact with the reference distributions. As one's memory tend to shape the composition of the reference set, one could expect that, just like it was shown earlier, one's adaptive evaluation will be sensitive to one's memory processes.

Suppose that at time t_0, the individual's history of observations consists of a collection of $T+1$ contexts $S_{t_0}, S_{t_0-1}, \ldots, S_{t_0-t}, \ldots, S_{t_0-T}$ with perceived magnitude distributions $F_{t_0}, F_{t_0-1}, \ldots, F_{t_0-t}, \ldots, F_{t_0-T}$. Suppose the individual remembers (samples) observations from period $t_0 - t$ uniformly with the memory rate $\delta_t \in [0,1]$, i.e. any two contextual magnitudes $y, y' \in S_{t_0-t}$ have equal probabilities of being included in the (remembered) reference set S. Using the general discounting model of Rubinstein [2003] (which permits hyperbolic discounting), write the remembered adaptive evaluation $I_S(x)$ of a magnitude x as a mixture of $T+1$ perceived probability distributions:

$$I_S(x) = \frac{\delta_0}{\delta_0 + \Delta} F_{t_0}(x) + \frac{1}{\delta_0 + \Delta} \sum_{t=1,\dots,T} \left(\Pi_{s=1,\dots,t}\delta_s\right) F_{t_0-t}(x) \tag{8}$$

where $\Delta = \sum_{t=1,\dots,T} \left(\Pi_{s=1,\dots,t}\delta_s\right)$.

Since past history may include imaginary environments, the above general formulation permits for saliency both in specific past histories and specific values. Specifically, a salient magnitude value $\tilde{x}$ might enter the reference set S as a past environment S_l containing a degenerate random variable at $\tilde{x}$, i.e. $F_l(x) = H(x - \tilde{x})$; while a salient environment at time k might enter S with $\delta_k > \max\{\delta_0, \dots, \delta_t, \dots, \delta_T\}$ for all $t \neq k$.

One can decompose the reference set S into a subset S_0 sampled from observations from the current period and a subset S_T sampled from remembered observations in the previous T periods, i.e. $S = S_0 \cup S_T$. Thus, the evaluation (8) can be written as a weighted sum of current F_{t_0} and remembered past $\mathcal{F}_T$ environment:

$$I_S(x) = \frac{\delta_0}{\delta_0 + \Delta} F_{t_0}(x) + \left(1 - \frac{\delta_0}{\delta_0 + \Delta}\right) F_T(x) \tag{9}$$

One can observe that memory determines whether one's evaluation exhibits context effects. Specifically, neoclassical context-independent utility function arises as a *special case* of adaptive evaluation when an individual has long and strong memory. This is because if one's evaluations are (almost) entirely determined by the past environment, present context has negligible effect on one's evaluations. In contrast, regardless of one's past experiences, as long as one's present has non-negligible effect, one's evaluation will exhibit context effects.

Proposition 7 *Suppose one's magnitude evaluation is determined by the past and present environments F_{t_0} and F_T as given by (9). Then*
(i) if $\delta_t > 0$ for all t and $T \to \infty$, then $I(x) = \lim_{T\to\infty} I_S(x)$ is independent of the current context.
(ii) if $\delta_0 > 0$, $T < \infty$, and S_T fixed, and an individual faces either of the two current environments, A or B with $F^A_{t_0}$ and $F^B_{t_0}$, respectively. Then $I_{S_T \cup A}(x) - I_{S_T \cup B}(x) = F^A_{t_0} - F^B_{t_0}$.
(ii) Suppose both cognitive tools are perfect, $p = 1$, and the sample size N is large, and the reference set S includes observations beyond the current environment with $\Delta > 0$. Then the adaptive evaluation I^P_S is suboptimal in a sense of Robson [2001].

Proof: (i) As $\lim_{T\to\infty} \Delta = \infty$, the weight of the current context is negligible: $\lim_{T\to\infty} \frac{\delta_0}{\delta_0+\Delta} = 0$ and thus $I(x) = \lim_{T\to\infty} I_S(x) = F_T(x)$.

(ii) Note that, given that S_T is fixed, $I_{S_T \cup A}(x) - I_{S_T \cup B}(x) = \frac{\delta_0}{\delta_0 + \Delta}(F^A_{t_0} - F^B_{t_0})$. Since $\frac{\delta_0}{\delta_0 + \Delta} > 0$, the rest is straightforward.
(ii) Since the reference set S includes observations beyond the current environment, $\frac{\delta_0}{\delta_0 + \Delta} < 1$, and thus $I_S(x) \neq F_{t_0}(x)$, and $I_S(x)$ is Robson-suboptimal. ∎

Thus, with long memory, the individual behaves as if he possesses a (neoclassical) context-independent utility function which is determined by the distribution of remembered past observations. Such context-independent evaluation, however, is not optimal in the sense of Robson [2001]. Conversely, if one's memory is bounded, one's evaluations will exhibit dependence on the current context. In other words, when an individual is presented with variable context, her magnitude evaluation will exhibit qualitative, but not quantitative, correspondence with the present context (e.g. Parducci [1963, 1965]). Furthermore, even though the conditions of Proposition 2 are satisfied, when an individual evaluates prospects based on a past memory, his adaptive evaluation differs from the current magnitude distribution, and thus his mistakes in a binary choice in the present environment S_{t_0} are not minimized.

2.7 Adaptive Evaluation of Economic Goods

Understanding the role of cognitive imperfections could be particularly relevant in a private ownership economies where most (if not all) ordinal comparisons are necessarily done interpersonally, particularly for economically relevant variables such as consumption, income, and wealth. To evaluate a coconut, Robinson Crusoe compares it against other coconuts present - whether the other coconuts are on a tree, on a store display, or in hands of other people. From the cognitive point of view, Mrs. Brown evaluates her house the same way as Crusoe evaluates a coconut, - using two primitive cognitive tools over a reference set consisting of other houses, - yet these reference houses tend to belong to other people. A change in Mr. Jones' possessions changes how Mrs. Brown values what she has, appearing as a desire to "keep up with the Joneses". While the adaptive evaluation is sentiment-free, in private ownership economies it may appear as a concern with social status discussed by Veblen [1899], Frank [1995], and many others. The same cognitive process involves evaluation of individual characteristics such as height, beauty, intelligence, and skill. When other comparison objects belong to other people, ordinal comparisons may be especially inaccurate, affecting one's evaluation of their possessions, with further economy-wide welfare implications.

Consider a private ownership economy, where each of N individuals is endowed with an object of magnitude x (e.g. height, intelligence, income, happiness, and so on). Suppose that these magnitudes of individual endowments are observable and thus constitute the reference set S with the veridical magnitude distribution F. Assume that all individuals have the same reference set S, have the same perfect frequency tool F, and the same ordinal tool.

First, observe that, as Proposition 2 suggests, if ordinal tool is perfect and if, whenever one sees other people's possessions to be equal to one's own, one judges them to be "smaller", one's evaluation will be optimal in a sense of Robson [2001]. This means, however, that the individual at the bottom of human hierarchy will have the lowest possible utility level, and the individual at the top will have the highest possible utility.

Yet, as Proposition 3 suggests, if all very small possessions appear to be similar, the bottom individual will be "happier" with own lot than what would be optimal. In contrast, if all very large possessions appear to be similar, the top individual will be less happy than it would be optimal. Thus, ordinal imperfections may "flatten" perceived value of possessions, reducing the potential disparity between those at the top and the bottom. It is thus possible that the way individuals compare themselves to the others may affect the welfare of the entire society. The utilitarian economic welfare in a society S_N can be written as

$$W_{S_N}(S_N) = \sum_{x \in S_N} I_{S_N}(x) \frac{N_{y=x}}{N_{y \in S_N}} \tag{10}$$

Observe however that the welfare can also be written as

$$W_{S_N}(S_N) = \sum_{i=1,\ldots,J} I_i f_i = \sum_{i=1,\ldots,J} \frac{\Sigma_{i-k-1} + p\hat{\Sigma}_{i-k:i}}{\Sigma_J} \frac{\hat{\Sigma}_{i-k:i}}{\Sigma_J}$$

which is equivalent to equation (6) for the context evaluation. In other words, the fundamental property of adaptive evaluation, stated in Proposition 5 can be restated as *the welfare neutrality of economic processes*. That is, utilitarian welfare is independent of economic processes (such as economics growth, redistribution, etc.) if and only if interpersonal comparison involves fair judgment, with half of all similar possessions being judged to be smaller and the other half being judged to be larger. However, once interpersonal comparisons arising from cognitive imperfections are not fair, utilitarian welfare depends on economic processes. The next example highlights the interaction between cognitive imperfections and inequality.

Example 7 *Consider an economy where proportion α of population have income x_L and proportion $1-\alpha$ have incomes $x_H > x_L$. Suppose all agents have the same ordinal tool which has the following property: whenever an individual endowed with x encounters another individual with endowment y, he perceived that $x \sim y$ whenever $\frac{x}{y} \in [\lambda^{-1}, \lambda]$, where $\lambda > 1$. Consider income distribution F_A which is sufficiently unequal with $x_H^A > \lambda x_L^A$. In this case, $I_A(x_L) = \alpha p$ and $I_A(x_H) = \alpha + p(1-\alpha)$, and welfare $W_A = p + (1-2p)\alpha(1-\alpha)$. Now consider instead income distribution F_B which is sufficiently equal with $x_H^B \leq \lambda x_L^B$. Then $I_B(x_L) = I_B(x_H) = p$, and $W_B = p$. As $W_A - W_B = (1-2p)\alpha(1-\alpha)$, welfare is higher in more equal society B as long as encounters with people of similar incomes tend to be seen favorably (i.e. $p > 0.5$). And vice versa, welfare is higher in more unequal society A if, instead, there is bias against encounters with people of similar incomes (i.e. $p < 0.5$). Notice that welfare neutrality holds whenever $p = 0.5$.*

Thus, despite its conceptual simplicity, the adaptive evaluation algorithm results in complex and unexpected interactions between social comparison processes and societal welfare.

2.8 Adding Frequency Imperfections

As Kahneman and Tversky [1979] point out, humans assign non-linear weights to probabilities, overweighing small probabilities and underweighing large ones.[7] Thus, it is plausible that the frequency tool is also imperfect, further distorting the relationship between the veridical magnitude distribution and adaptive evaluation, and making it more difficult to observe empirically. Hsu, Krajbich, Zhao and Camerer [2009] found that the neural activity in the striatum is nonlinear in stimulus probabilities, suggesting a possibility that the probability distortions may arise at the level of coding in the brain, while Stewart, Chater and Brown [2006] noted that probability distortions align with the empirical frequency distributions. However, the exact form and mechanism behind probability imperfections is still unclear, as recent research suggests that experienced and stated probabilities may result in opposite pattern of overweighting and underweighting of small and large probabilities (Barron and Erev [2003], Hertwig, Barron, Weber, and Erev [2004]), while Harbaugh, Krause, and Vesterlund [2003, 2010] further report that the implied shape of subjects' probability weighting function is affected by the elicitation procedure and age, with a possibility of the probability weighting function to be linear in adults.

Psychologists Spence [1990] and Hollands and Dyre [2000] found even a more complex phenomenon for proportion judgments over a variety of stimuli, including continuous sensory modalities such as probabilities, proportions of graphical elements, numerosities, fullness of the glass, and so on). Such judgments are found to exhibit non-linearities, involving relatively accurate evaluations at the "intermediate" proportions, overestimation of low and underestimate of large proportions

Whatever is the form of frequency (proportion) imperfections, one can build them into the adaptive evaluation algorithm and generalize the realized magnitude evaluation (2) as:

$$\hat{I^{\Upsilon}}_{S_N}(x) = \Upsilon\left(\frac{\#(y \in \hat{S}_{\prec}(x))}{\#(y \in S)}\right) = \Upsilon\left(\frac{\#(S_{y \prec x} \cup S^{\prec}_{y \sim x})}{\#(y \in S)}\right) \tag{11}$$

Here, $\Upsilon(x)$ is the *perceived reference distribution* $\Upsilon(x)$ representing a perceived probability that a magnitude x is judged not exceed an arbitrary other reference magnitude y. The formulation (11) takes into account that the frequency imperfections for discrete proportions of countable objects may differ from those for continuous proportions of continuous modalities such as luminosities.

Again, since the ordinal comparison tool is stochastic, so is the *realized evaluation* $\hat{I^{\Upsilon}}_{S_N}(x)$, and thus, the realized evaluation $\hat{I^{\Upsilon}}_{S_N}$ can be decomposed into its expectation, which will be called the *adaptive evaluation* $I^{\Upsilon}_{S_N}$, and a random noise variable ϵ:

$$\hat{I^{\Upsilon}}_{S_N} = I^{\Upsilon}_{S_N} + \epsilon$$

[7] See Hsu, Krajbich, Zhao and Camerer [2009] for a comprehensive list of probability weighting models.

While the realized evaluations $\hat{I}^{\Upsilon}(x)$ are generated by a binomial process, their numerical values tend to be affected non-linearly by the frequency imperfections, resulting in the adaptive evaluation with frequency imperfections to be functionally different from the one with perfect frequency tool.

The research on probability/proportion imperfections is still in infancy, and the mechanism behind frequency distortions is not clear. Interestingly, Dehaene, Izard, Spelke, and Pica [2008] found that subjects tend to undercount large numerosities - particularly for sets of dots and sequences of tones, - and such mistakes are more pronounced in indigenous societies.[8] Thus it is possible that, in large reference sets, the total number of elements N might be undercounted, as well as the number of smaller items $N_{y<x}$ when magnitude x is large (as it "beats" many reference magnitudes). Such *counting errors*, just like ordinal imperfections, may lead to overestimation of cumulative frequency for small magnitudes and underestimation for large ones.

Example 8 *Consider the context A of Example 1. For illustrative purposes, suppose numerosities greater than* 6 *are undercounted by* 1*. With the perfect ordinal tool, the realized evaluations are:*

$$\begin{array}{ll}
\hat{I}^{\Upsilon}{}_A(10) \in \left\{\frac{0}{10-1}; \frac{1}{10-1}\right\} & \hat{I}^{\Upsilon}{}_A(18) \in \left\{\frac{4}{10-1}; \frac{5}{10-1}; \frac{6}{10-1}; \frac{7-1}{10-1}; \frac{8-1}{10-1}\right\} \\
\hat{I}^{\Upsilon}{}_A(11) \in \left\{\frac{1}{10-1}; \frac{2}{10-1}\right\} & \hat{I}^{\Upsilon}{}_A(19) \in \left\{\frac{8-1}{10-1}; \frac{9-1}{10-1}\right\} \\
\hat{I}^{\Upsilon}{}_A(12) \in \left\{\frac{2}{10-1}; \frac{3}{10-1}; \frac{4}{10-1}\right\} & \hat{I}^{\Upsilon}{}_A(20) \in \left\{\frac{9-1}{10-1}; \frac{10-1}{10-1}\right\}
\end{array}$$

with adaptive evaluations being:[9]

$$\begin{array}{lll}
I^{\Upsilon}_A(10) = \frac{p}{9} & I^{\Upsilon}_A(11) = \frac{1+p}{9} & I^{\Upsilon}_A(12) = \frac{2+2p}{9} \\
I^{\Upsilon}_A(18) = \frac{4+4p-4p^3+3p^4}{9} & I^{\Upsilon}_A(19) = \frac{7+p}{9} & I^{\Upsilon}_A(20) = \frac{8+p}{9}
\end{array}$$

*Just like ordinal imperfections of the Example 2, numerosity miscounts here lead to overestimation of cumulative frequency for small magnitudes (*10 *and* 11*) and underestimation for large ones (*19 *and* 20*). The context evaluation is further affected by the frequency miscounts:*

$$E^{\Upsilon}_A(A) = \frac{p}{9}\frac{1}{10-1} + \frac{1+p}{9}\frac{1}{10-1} + \frac{2+2p}{9}\frac{2}{10-1} + \frac{4+4p-4p^3+3p^4}{9}\frac{4}{10-1} + \\ + \frac{7+p}{9}\frac{1}{10-1} + \frac{8+p}{9}\frac{1}{10-1} = \frac{36+24p-16p^3+12p^4}{81}$$

With frequency imperfections, the fundamental property fails, as $E^{\Upsilon}_A(A) = \frac{187}{324} > \frac{1}{2}$ for $p = \frac{1}{2}$.

[8] Furthermore, in Dehaene, Izard, Spelke, and Pica [2008] subjects from Amazonian indigenous societies also appear to overcount small numerosities, which might further lead to overcounting the number of smaller items $N_{y<x}$ when magnitude x is small (as it "beats" only a few reference magnitudes).

[9] Since there are $n = 4$ elements with veridical magnitude 18, the realized evaluations $\hat{I}^{\Upsilon}{}_A(18)$ are binomially distributed with $\frac{n!}{k!(n-k)!}p^k(1-p)^{n-k}$. However, the counting errors result in $\hat{I}^{\Upsilon}{}_A(18) = \frac{6}{9}$ appearing more frequently, leading to a complex expression for the adaptive evaluation $I^{\Upsilon}_A(18)$.

While the ordinal imperfections coarsen the reference set (as in the Example 2), their effect is further distorted by the frequency miscounts. To see that, observe first that with the combined ordinal and frequency imperfections, the realized evaluations are

$$\hat{I}^{\Upsilon}_{A'}(x) \in \left\{ \frac{0}{10-1}; \frac{1}{10-1}; \frac{2}{10-1}; \frac{3}{10-1}; \frac{4}{10-1} \right\} \quad \textit{for} \quad x \in \{10, 11, 12\}$$

$$\hat{I}^{\Upsilon}_{A'}(x) \in \left\{ \frac{4}{10-1}; \frac{5}{10-1}; \frac{6}{10-1}; \frac{7-1}{10-1}; \frac{8-1}{10-1}; \frac{9-1}{10-1}; \frac{10-1}{10-1} \right\} \quad \textit{for} \quad x \in \{18, 19, 20\}$$

so that the adaptive evaluations are:

$$\begin{aligned} I^{\Upsilon}_{A'}(10) = I^{\Upsilon}_{A'}(11) = I^{\Upsilon}_{A'}(12) &= \frac{4p}{9} \\ I^{\Upsilon}_{A'}(18) = I^{\Upsilon}_{A'}(19) = I^{\Upsilon}_{A'}(20) &= \frac{4 + 6p - 20p^3 + 45p^4 - 36p^5 + 10p^6}{9} \end{aligned}$$

with overvaluation of the small magnitudes and overvaluation of large magnitudes relatively to their veridical values of the Example 1. The context evaluation is

$$\begin{aligned} E^{\Upsilon}_{A'}(A') &= \frac{4p}{9}\frac{4}{10-1} + \frac{4 + 6p - 20p^3 + 45p^4 - 36p^5 + 10p^6}{9}\frac{6}{10-1} = \\ &= \frac{24 + 52p - 120p^3 + 270p^4 - 216p^5 + 60p^6}{81} \end{aligned}$$

Again, the fundamental property fails, as $E^{\Upsilon}_{A'}(A') = \frac{779}{1296} > \frac{1}{2}$.

Thus, when the ordinal and frequency imperfections are combined, frequency imperfections may aggravate the effects of the ordinal imperfections of small magnitude evaluations and may not be sufficient to counteract the effects of ordinal imperfections on the evaluations of large magnitudes. Importantly, as the above Example shows, the fundamental property of adaptive evaluation tends to fail in the presence of frequency imperfections. This also highlights a possibility that frequency imperfections might play a further role in assessing one's economic well-being and societal welfare.

3 Adaptive Magnitude Evaluation: Continuous Case

Despite its conceptual simplicity, the adaptive evaluation model requires non-trivial mathematical treatment. The use of convolution/scale mixture technics for continuous reference distributions allows to push the analysis further.

3.1 Convolution/Scale Mixture Model

Let the environment S be represented by a veridical (non-degenerate) distribution F, continuously differentiable on support $(a,b)\subseteq R$, with $f=F'>0$. Let cumulative density function $F(x)$ be the *frequency tool*, as $F(x) = Pr(y \leq x)$, or the frequency with which y is less or equal than x. As the *ordinal comparison tool*, let the expected outcome of pairwise comparison of a fixed magnitude x with any magnitude y be represented by the *pairwise comparison function* $\mathcal{D}(x,y): S \rightarrow [0,1]$, non-decreasing in x, non-increasing in y, and discontinuous at a countable number of points. Thus, for any magnitude $x \in S$, the adaptive evaluation $I_S(x)$ of x using two basic cognitive tools on the reference set S is given by the expected perceived outcome of pairwise comparison tournament of x against every element y in reference set S, or the perceived rank of x in S:[10]

$$I_S(x) = \int_S \underbrace{\mathcal{D}(x,y)}_{ordinal\ tool} \underbrace{dF(y)}_{frequency\ tool} \tag{12}$$

Proposition 8 *Suppose an individual is endowed with a frequency (proportion) tool F and a pairwise comparison tool $\mathcal{D}(x,y)$, which is non-decreasing in x, non-increasing in y, and has a finite number of discontinuities. Then*

(i) $I_S(x)$ is a non-decreasing continuous function on set S, and thus rationalizes continuous "more is better" preferences.

(ii) if set A stochastically dominates set B, i.e. $F_A \succeq_{FOSD} F_B$, then $I_B(x) \geq I_A(x)$ for any $x \in X$.

Proof: (i) Since $\mathcal{D}$ is increasing in x and is discontinuous at finitely many points, $I_S(x)$ is non-decreasing and continuous. The preference rationalizability is a standard result.

(ii) By definition of first order stochastic dominance, for any non-decreasing function $U(t)$, $\int_A U(t)dF_A(t) \geq \int_B U(t)dF_B(t)$. But since $\mathcal{D}$ is non-increasing in y, we have that

$$I_A(x) = \int_A \mathcal{D}(x,y)dF_A(y) \leq \int_B \mathcal{D}(x,y)dF_B(y) = I_B(x) \quad \blacksquare$$

Thus, the adaptive evaluation $I_S(x)$ can rationalize continuous more is better preferences on S. Yet once the context changes, so does the adaptive evaluation. Thus, both reflexivity and transitivity may be violated, as one can find some $\beta \in A \cap B$, such that $I_A(\beta) \neq I_B(\beta)$,

[10] Formally, the adaptive evaluation model (12) is closely related to the "expected utility without utility" of Castagnoli and LiCalzi [1996], who suggested to compare lotteries by the expectation of one lottery outperforming the other.

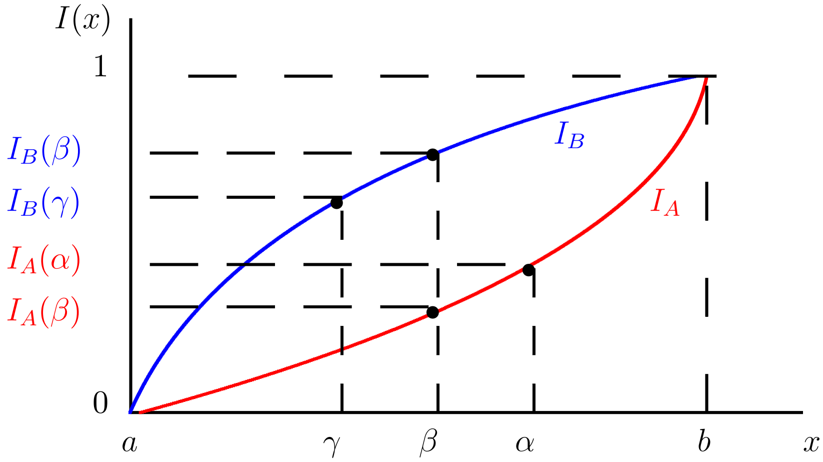


Figure 1: Violations of reflexivity and transitivity across different contexts for $F_A \succeq_{FOSD} F_B$.

and it is possible to find $\alpha \in A$ and $\gamma \in B$ such that $I_A(\alpha) > I_A(\beta)$ and $I_B(\beta) > I_B(\gamma)$, but $I_A(\alpha) < I_B(\gamma)$ (see Figure 1). This has an important consequence. Suppose there exist an "outside" object with exogenously determined valuation $I_Y(\delta)$ where $(A \cup B) \cap Y = \emptyset$ such that $I_A(\beta) < I_Y(\delta) < I_B(\beta)$. Then, an individual would trade β for δ when reference set is A, but once reference set changes to B, he would trade δ back for β - an apparent preference reversal.

Suppose the outcome of pairwise comparison between x and y is determined with certainty whenever x and y are sufficiently far apart, but the ordinal discrimination is stochastic whenever x and y are sufficiently close. Suppose the ordinal discrimination depends the ordinal discriminability variable $z = Z(x, y)$ with $Z_1 > 0, Z_2 < 0$. Let $G(z)$ be the probability that x is perceived to be no smaller when compared to y whenever z falls into the *interval of doubt* $[k_1, k_2]$, with $g(z) = \frac{dG}{dz} > 0$ for all $z \in [k_1, k_2]$. Then, the outcome of the imperfect ordinal comparison tournament is:

$$\mathcal{D}^Z(x, y) = \begin{cases} 0 & \text{whenever} \quad z < k_1 \\ G(z) & \text{whenever} \quad z \in [k_1, k_2] \\ 1 & \text{whenever} \quad z > k_2 \end{cases} \tag{13}$$

and the imperfect adaptive evaluation $I_S(x)$ can be written as the expected outcome of pairwise comparison:

$$I_S^Z(x) = \int_S \mathcal{D}^Z(x, y) dF(y) = \int_S G(Z(x, y)) dF(y) \tag{14}$$

The imperfect adaptive evaluation $I_S^Z(x)$ depends on the form of discriminability $z = Z(x, y)$. Here, I utilize the isomorphism of between the expected outcome of imperfect pairwise comparisons and algebra of random variables, and use the convolution and scale mixture techniques to model the additive and multiplicative ordinal imperfections (see, for example, Springer [1979]). Thus I concentrate on two general forms of ordinal imperfections - additively imperfect ordinal comparison with $z = x+y$ and multiplicatively imperfect ordinal comparison with $z = \frac{x}{y}$). These two forms of imperfect evaluations have many features in common - at least in part because one can use a transformation $G\left(\frac{x}{y}\right) = \tilde{G}\left(\ln\frac{x}{y}\right) = \tilde{G}(\tilde{x} - \tilde{y})$, where $\tilde{x} = \ln x$ and $\tilde{y} = \ln y$. While additively imperfect ordinal comparison tool with $z = x + y$ is computationally simpler and thus potentially easier to understand, multiplicatively imperfect tool with $z = \frac{x}{y}$ is more conducive to cognitive non-linearities and has a more solid grounding in cognitive and brain research.

3.2 The Benchmark Model of "Perfect" Adaptive Evaluation

Adaptive evaluation with perfect cognitive tools provides a useful benchmark for understanding the role of cognitive imperfections.

Suppose an individual is endowed with a *perfect* frequency (or proportion) tool, so that he correctly perceives the veridical distribution of magnitudes F. Suppose further he is endowed with a *perfect* ordinal comparison tool, i.e. he can tell a bigger from a smaller magnitude even if the magnitudes are only slightly different. In this case, an expected outcome of a pairwise comparison can be written as:[11]

$$\mathcal{D}^P(x,y) = \begin{cases} 0 & \text{whenever} \quad x < y \\ \frac{1}{2} & \text{whenever} \quad x = y \\ 1 & \text{whenever} \quad x > y \end{cases} \tag{15}$$

This perfect ordinal comparison tool permits two isomorphic representations as a degenerate discriminability z. First, perfect ordinal tool $\mathcal{D}^P$ can be represented in terms of a difference $z = x - y$, degenerate at 0, with $G(z) = H(z) = H(x - y)$, where $H(\cdot)$ is Heaviside (step) function.[12] Thus, the perfect ordinal tool admits additive representation as

$$\mathcal{D}^P(x,y) = H(x - y) \tag{16}$$

Second, $\mathcal{D}^P$ can be represented in terms of a ratio $z = \frac{x}{y}$, degenerate at 1, with $G(z) = H(z-1) = H\left(\frac{x}{y} - 1\right)$. Thus, the multiplicative representation of the perfect ordinal tool is

$$\mathcal{D}^P(x,y) = H\left(\frac{x}{y} - 1\right) \tag{17}$$

[11] This perfect ordinal tool is isomorphic to the optimal happiness function of Rayo and Becker [2007], who model perceptual limitations involved when the expected happiness is compared across different choice options, rather than built into the two cognitive tools considered here.

[12] A degenerate random variable z at c has a cumulative distribution equal to Heaviside (step) function $H(z-c)$, with $H(z-c) = 0$ for $z < c$, $H(z-c) = 1$ for $z > c$, and $H(c) \in [0,1]$. It has Dirak delta density function $\delta(x-c)$, and domain $(-\infty, \infty)$. For an arbitrary distribution F, $\int_{-\infty}^{\infty} H(c-z)dF(z) = F(c)$.

Since $H\left(\frac{x}{y}-1\right) = H(x-y)$, the formulations (16) and (17) represent the same degenerate random variable. This allows us to derive the following benchmark result.

Proposition 9 *If an individual is endowed with a perfect ordinal comparison tool (15) and with a perfect frequency (proportion) tool F, then his adaptive evaluation of magnitude x is given by the cumulative density function of the veridical (true) magnitude distribution $F(x)$:*

$$I_S^P(x) = F(x) \tag{18}$$

Moreover, the perfect adaptive evaluation is optimal in the sense of Robson [2001].

Proof: Using representations (16) and (17), get

$$I_S^P(x) = \int_S \mathcal{D}^P(x,y)dF(y) = \int_S H(x-y)\,dF(y) = \int_S H\left(\frac{x}{y}-1\right)dF(y) = F(x)$$

The optimality follows from Robson [2001] and Netzer [2009]. ∎

That is, if both ordinal comparison and frequency (proportion) tools are perfectly accurate, an individual's evaluation $I_S^P(x)$ of magnitude x is isomorphic to the veridical cumulative density function (veridical rank) $F(x)$, and is equal to the frequency of a magnitude x ordinarily "outperforming" other elements in the reference set S. Moreover, as it was shown by Robson [2001] and Netzer [2009], if the adaptive utility is $F(x)$, the probability of mistake in a binary choice in an environment S is minimized. As the rest of the paper will show, cognitive imperfections result in an adaptive evaluation which is functionally distinct from the veridical distribution F, and thus suboptimal in the sense of Robson [2001].

When cognitive tools are perfect, the shape of the magnitude distribution entirely determines the shape of the utility function. For example, if the veridical magnitude distribution is of the power function form, individual's adaptive evaluation is consistent with a constant relative risk aversion (CRRA) utility function; and if the veridical distribution is exponential, the evaluation exhibits constant absolute risk aversion (CARA).[13]

3.3 Context Evaluation with Perfect Tools

The above cognitive evaluation model allows one to evaluate magnitude x based on one's environment S. One can make a step further and evaluate the entire context S as a gamble by utilizing Gilboa and Schmeider [2003]'s suggestion that one could use one's environment S to extract probability information.

[13] See Castagnoli and Li Calzi [1996] for these relationships between distribution functions and the shape of utility function.

Consider an environment S with magnitude distribution F_S with support (a, b). The first result shows that, when both cognitive tools are perfect and all observations in the database are relevant both for magnitude evaluation and for probability calculation, the evaluation of a stochastic environmental context is independent of that context.

Proposition 10 *Consider an environment S with magnitude distribution F_S with support (a, b). If the cognitive tools are perfect, the context evaluation EI_S^P is independent of the environment S and is equal to the evaluation of the median magnitude.*

Proof: The proof is simple:

$$EI_S^P(S) = \int_S I_S^P(x)dF_S(x) = \int_S F_S(x)dF_S(x) = \frac{1}{2} \quad \blacksquare$$

In other words, if one faces a single stochastic context, one's evaluation of that gamble is equivalent to the evaluation of the median magnitude. When distribution is positively skewed (with frequent smaller magnitudes), its median magnitude is smaller than mean magnitude, and the adaptive evaluation of such context is lower than the evaluation of the average magnitude, leading to apparent *risk aversion* (as the expected value of a gamble is less than the value of its expectation). In contrast, if the distribution is negatively skewed (with frequent larger magnitudes), the expected value of the gamble exceeds the value of its expectation, corresponding to *risk loving*. We thus have the following straightforward result.

Corollary 3 *Consider an environment S with magnitude distribution F_S with support (a, b), and average magnitude $\mu = E(x)$ and median $m = F_S^{-1}(0.5)$. If the cognitive tools are perfect, then $I_S E(x) > (<) EI_S(S)$ iff $\mu > (<) m$.*

That is, one's choice between a gamble and its expectation could potentially be governed solely by the properties of the stochastic environment. However, the memory of previously encountered environments could be important for behavior in risky situations. To see that, suppose there are two environments A and B, with F_A and F_B on (a, b). If one's memory can only hold information about one environment only, then Proposition 10 would apply, as it holds for any environment S.

Instead, suppose the individual has perfect memory, and able to associate each reference element with the corresponding environment. For the purposes of magnitude evaluation, one uses the "global" reference set $A \cup B$:

$$I_{A\cup B}(x) = \int_{A\cup B} \mathcal{D}^P(y, x)(\alpha f_A(y) + (1-\alpha) f_B(y))dy = \alpha F_A(x) + (1-\alpha)F_B(x)$$

where α and $1-\alpha$ denote the shares of elements belonging to environments A and B, respectively. However, the probability distributions associated with each context are still F_A and F_B.[14] Thus, the context evaluation of each environment $j = A, B$ given the combined reference set $A \cup B$ are:

$$\begin{aligned} EI_{A\cup B}(A) &= \int_{A\cup B} (\alpha F_A(x) + (1-\alpha)F_B(x))dF_A(x) = \frac{\alpha}{2} + (1-\alpha)\int_{A\cup B} F_B(x)dF_A(x) \\ EI_{A\cup B}(B) &= \int_{A\cup B} (\alpha F_A(x) + (1-\alpha)F_B(x))dF_B(x) = \frac{1-\alpha}{2} + \alpha\int_{A\cup B} F_A(x)dF_B(x) \end{aligned}$$

In other words, the distribution of elements in context B affects the expected magnitude evaluation of context A. One thus would expect the the choice of one gamble over the other is affected by the interaction between the two contexts, as the following proposition suggests.

Proposition 11 *Consider two environments A and B with magnitude distributions F_A and F_B with support (a, b). If the agent's cognitive tools are perfect, then $A \succ (\prec) B$ whenever $\int_{A\cup B} F_B(x)dF_A(x) > (<)\frac{1}{2} \Leftrightarrow \int_{A\cup B} F_A(x)dF_B(x) < (>)\frac{1}{2}$. Moreover, unless $F_A = F_B$, indifference is non-generic.*

Proof: The agent's choice depends on the difference between the two expectations:

$$EI_{A\cup B}(A) - EI_{A\cup B}(B) = \frac{\alpha}{2} + \int_{A\cup B} (1-\alpha)F_B(x)dF_A(x) - \frac{1-\alpha}{2} - \int_{A\cup B} \alpha F_A(x)dF_B(x)$$

Note that $\int_{A\cup B} F_A(x)dF_B(x) + \int_{A\cup B} F_B(x)dF_A(x) = 1$, so that

$$EI_{A\cup B}(A) - EI_{A\cup B}(B) = \int_{A\cup B} F_B(x)dF_A(x) - \frac{1}{2} = \frac{1}{2} - \int_{A\cup B} F_A(x)dF_B(x) \blacksquare \tag{19}$$

As the next result states, exposing one to information about an alternative context makes one better off if the alternative context is stochastically worse, and vice versa. Moreover, as one would expect, the stochastically bigger context is more desirable.

Corollary 4 *Iff $F_A \succeq_{FOSD} F_B$, then $EI_{A\cup B}(A) \geq EI_A(A)$, $EI_{A\cup B}(B) \leq EI_B(B)$, and $A \succeq B$.*

Proof: First order stochastic dominance implies that $F_B(x) \geq F_A(x)$ for all x on (a, b), thus

$$\begin{aligned} \int_{A\cup B} F_B(x)dF_A(x) &\geq \int_{A\cup B} F_A(x)dF_A(x) = \frac{1}{2} \\ \int_{A\cup B} F_A(x)dF_B(x) &\leq \int_{A\cup B} F_B(x)dF_B(x) = \frac{1}{2} \end{aligned}$$

[14] This partitioning the memory set into disjoint subsets for the purpose of probability calculation can be modeled by the similarity function of Gilboa and Schmeider [1995, 2003].

Thus, $EI_{A\cup B}(A) \geq EI_A(A) = \frac{1}{2}$ and $EI_{A\cup B}(B) \leq EI_B(B) = \frac{1}{2}$, and $EI_{A\cup B}(A) \geq EI_{A\cup B}(B)$. ∎

The next result assumes that the distribution F_B is concave, so that bigger magnitudes are less frequent than smaller ones. If, in addition, it is second order stochastically dominated by the other context F_A, then F_B is less desirable. This happens for two reasons - first, because from the magnitude evaluation point of view, the frequent small magnitudes in B make magnitudes in A to be more attractive, and, second, because bigger magnitudes are more frequent in A than in B.

Corollary 5 *If $F_A \succeq_{SOSD} F_B$ and $F_B'' < 0$ on (a, b), then $EI_{A\cup B}(A) \geq EI_A(A)$, and $A \succeq B$.*

Proof: As F_B is increasing and concave, second order stochastic dominance implies that

$$EI_{A\cup B}(A) - EI_{A\cup B}(B) = \int_{A\cup B} F_B(x)dF_A(x) - \frac{1}{2} \geq \int_{A\cup B} F_B(x)dF_B(x) - \frac{1}{2} = 0 \ ∎$$

The context evaluation generates the same prediction as expected utility theory in the case of first order stochastic dominance, but is more restrictive in the case of second order stochastic dominance. Yet, as Stewart, Chater and Brown [2006] point out, empirically observed distributions tend to exhibit everywhere decreasing density. Thus, from the empirical point of view, the cases when the expected utility theory and the theory of adaptive magnitude evaluation generate different predictions are not common.

3.4 Additively Imperfect Ordinal Comparison Tool

Let the ordinal discriminability z be in terms of differences, i.e. $z = x - y \in [k_1, k_2]$, with $k_1 \leq 0 \leq k_2$. For technical simplicity, assume that the interval of doubt is "small", i.e. $0 < k_2 - k_1 \leq b - a$. Whenever $z > 0$, a reference magnitude y "looms large", with the ordinal tournament assessment of magnitude x being biased downwards. And vice versa, whenever $z < 0$, a reference magnitude y "looms small", and the evaluation of x is boosted upwards. Furthermore, note that whenever $G(0) > 0.5$, one's "upward" comparisons are less accurate than "downward" comparisons, in a sense that one tends to incorrectly perceive more frequently "winning" to larger magnitudes than "losing" to smaller magnitudes, leading to "upward bias", and vice versa, with $G(0) = 0.5$ corresponding to "fair" (symmetrically inaccurate) ordinal imperfection.

The resulting adaptive evaluation $I_S(x)$ is isomorphic to a *convolution* of a reference variable y and an ordinal discriminability variable z:

$$I_S^A(x) = \int_S \mathcal{D}^A(x, y)dF(y) = \int_S G(x - y)\, dF(y) \tag{20}$$

with the marginal evaluation function being

$$\frac{dI_S^A(x)}{dx} = \int_S g\,(x-y)\, f(y)dy \quad \text{where} \quad x = z + y$$

Obviously, the perfect comparison tool (16) is a special case of (20). This isomorphism of the additively imperfect adaptive evaluation and a distribution of a variable $x = y + z$ simplifies the subsequent analysis.

Proposition 12 *Suppose the expectation of ordinal discriminability z is zero, i.e. $Ez = 0$. Then, the additively imperfect adaptive evaluation of $I^A(x)$ crosses the veridical adaptive evaluation $I^P(x)$ once, and from above.*

Proof: The reference magnitude Y second order stochastically dominates the additively imperfect reference magnitude $Y + Z$ (or $Y \geq_{SOSD} Y + Z$), and the result is straightforward (see Shaked and Shantikumar [2007], Theorems 3.A.5 and 3.A.34). ∎

In other words, whenever ordinal discriminability z has zero mean, the low values of x are adaptively overvalued, and the high values are undervalued. For example, if the veridical (true) reference distribution is normal $N(\mu_x, \sigma_x^2)$, and the ordinal discriminability variable z is also normal $N(0, \sigma_z^2)$, then the additively imperfect adaptive evaluation of x is the rank in the convolution distribution, which is normal $N(\mu_x, \sigma_x^2 + \sigma_z^2)$, having "fatter" tails than the veridical (true) distribution. This "fatter tail" result holds for most reference distributions.

Proposition 13 *Suppose an individual is endowed with a perfect frequency (proportion) tool F, and an additively imperfect ordinal comparison tool D^A (20) with the ordinal discriminability variable z distributed with G on $[k_1, k_2]$ with $k_1 < 0 < k_2$ and $k_2 - k_1 \leq b - a$. Then $I_S^A(a) > 0$ and $I_S^A(b) < 1$ as long as $g = G' > 0$ on $[k_1, k_2]$ and $a > -\infty$ and $b < \infty$.*

Proof: As $k_2 - k_1 \leq b - a$, an additively imperfect evaluation $I_S^A(x)$ is:

$$I_S^A(x) = \begin{cases} \int_a^x \int_a^{t-k_1} g(t-y)dF(y)dt & \text{if } a \leq x \leq a + k_2 \\ \int_{a+k_2}^x \int_{t-k_2}^{t-k_1} g(t-y)dF(y)dt \quad +I_S^A(a+k_2) & \text{if } a + k_2 < x < b + k_1 \\ \int_{b+k_1}^x \int_{t-k_2}^{b} g(t-y)dF(y)dt \quad +I_S^A(b+k_1) & \text{if } b + k_1 \leq x \leq b \end{cases} \tag{21}$$

Thus, $I_S^A(a) = \int_{a+k_1}^a \int_a^{t-k_1} g(t-y)dF(y)dt$ is strictly positive as long as $a > -\infty$, and $I_S^A(b) = 1 - \int_b^{b+k_2} \int_{t-k_2}^b g(t-y)dF(y)dt$ is strictly less than one as long as $b < \infty$. ∎

Thus, ordinal imperfection alter the shape of magnitude evaluation, obscuring the relationship between context and evaluation, as an imperfect additively imperfect evaluation may not be a distribution function. Ordinal imperfections tend to result in overvaluation of low magnitides and undervaluation of high magnitudes because, when evaluating small objects, the individual incorrectly perceives the existence of objects that are smaller than the smallest object in the set S, and, similarly, he imagines non-existing large objects when evaluating large magnitudes.

Example 9 *Suppose the reference distribution F is uniform on $[\mu-\sigma;\mu+\sigma]$, and distribution of ordinal discriminability $G(z)$ is also uniform on $[k_1, k_2]$ with $k_2 - k_1 \leq 2\sigma$. Then*

$$I_S^{AU}(x) = \begin{cases} \frac{(\mu-\sigma+k_1-x)^2}{4\sigma(k_2-k_1)} & \text{if } \mu-\sigma \leq x \leq \mu-\sigma+k_2 \\ \frac{1}{2} + \frac{x-\mu}{2\sigma} - \frac{k_1+k_2}{4\sigma} & \text{if } \mu-\sigma+k_2 < x < \mu+\sigma+k_1 \\ 1 - \frac{(\mu+\sigma+k_2-x)^2}{4\sigma(k_2-k_1)} & \text{if } \mu+\sigma+k_1 \leq x \leq \mu+\sigma \end{cases}$$

Thus, in addition to overvaluing small and undervaluing large magnitudes, the evaluation exhibits increasing marginal utility for small magnitudes, decreasing marginal utility for large magnitudes, and constant marginal utility in-between.

3.5 Multiplicatively Imperfect Ordinal Comparison Tool

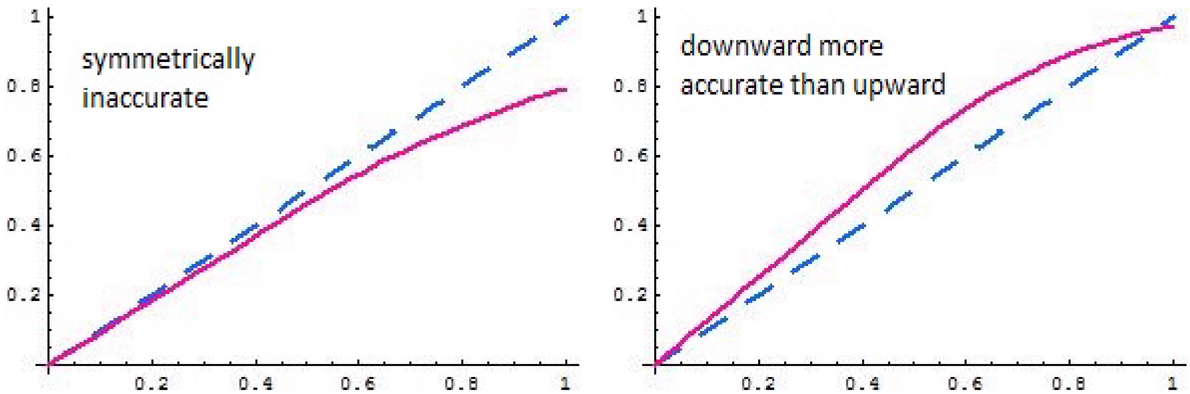
Let us consider a generalization of Rubinstein's [1988] similarity model (which can be seen as a form of psychophysical Weber-Fechner law). Let $a \geq 0$ and suppose the ordinal discriminability z be in terms of ratios, i.e. $z = \frac{x}{y} \in [k_1, k_2]$, with $0 < k_1 \leq 1 \leq k_2$. That is, the individual can tell apart two distinct magnitudes x and y with certainty only if the ratio of magnitudes $z = \frac{x}{y}$ is outside of the interval of doubt $[k_1, k_2]$, but within this interval an individual can only tell that x is greater than y with some probability $G(z)$. For technical simplicity, assume that the interval of doubt is "small", i.e. $1 < \frac{k_2}{k_1} \leq \frac{b}{a}$. Whenever $z > 1$, a reference magnitude y "looms large", with the ordinal judgment of magnitude x being biased downwards. And vice versa, whenever $z < 1$, a reference magnitude y "looms small", and the judgment of x is boosted upwards. Furthermore, note that whenever $G(1) > 0.5$, one's "upward" comparisons are less accurate than "downward" comparisons, in a sense that one tends to incorrectly perceive more frequently "winning" to larger magnitudes than "losing" to smaller magnitudes, leading to "upward bias", and vice versa, with $G(1) = 0.5$ corresponding to "fair" (symmetrically inaccurate) ordinal imperfection.

The resulting adaptive evaluation $I_S(x)$ is isomorphic to a *scale mixture* of a reference variable y and an ordinal discriminability variable z:

$$I_S^M(x) = \int_S \mathcal{D}^M(x,y)dF(y) = \int_S G\left(\frac{x}{y}\right) dF(y) \tag{22}$$

with the marginal evaluation function being

$$\frac{dI_S^M(x)}{dx} = \int_S \frac{1}{y} g\left(\frac{x}{y}\right) f(y)dy \quad \text{where} \quad x = z \cdot y$$



a) $x \sim U(0,1)$, $z \sim U[0.5, 2.0]$ b) $x \sim U(0,1)$, $z \sim U[0.5, 1.2]$

Figure 2: Relatively to the uniform veridical distribution on $(0, 1)$ (dashed lines) the multiplicatively imperfect evaluation (solid curves) exhibits non-increasing diminishing marginal utility.

Obviously, the perfect comparison tool (17) is a special case of (22).

The isomorphism of the multiplicatively imperfect adaptive evaluation and a distribution of a variable $x = y \cdot z$ leads to a number of interesting results, including to an interesting observation that with an *arbitrary* multiplicative discriminability, one's adaptive evaluation based on the uniform reference distribution on $(0, 1)$ will tend to exhibit non-increasing marginal evaluation.

Proposition 14 *Suppose the veridical distribution F is uniform on $(0, 1)$. Then, the multiplicatively imperfect ordinal tool results in a concave magnitude evaluation.*

Proof: This follows from the converse to Khinchine's representation for unimodal distributions and the definition of unimodal distributions (see Dharmadhikari and Joag-dev [1988], Theorem 1.3). ∎

For example, as Figure 2 shows, evaluation relatively to the uniform reference distribution on $(0, 1)$ with a uniform multiplicative tool, will exhibit constant marginal utility for the lower range of veridical magnitudes and decreasing marginal utility for the higher range. The following statement shows that the "fatter tail" result holds for the multiplicatively imperfect ordinal tool as well.

Proposition 15 *Suppose an individual is endowed with a perfect frequency (proportion) tool F, and a multiplicatively imperfect ordinal comparison tool D^M (22) with ordinal discriminability z distributed with G on $[k_1, k_2]$ with $0 < k_1 < 1 < k_2$ and $\frac{k_2}{k_1} \leq \frac{b}{a}$. Then $I_S^M(a) > 0$ and $I_S^M(b) < 1$ as long as $a > -\infty$ and $b < \infty$.*

Proof: For small interval of doubt $\frac{k_2}{k_1} < \frac{b}{a}$, adaptive evaluation $I_S^M(x)$ is:

$$I_S^M(x) = \begin{cases} \int_{k_1 a}^{x} \int_{a}^{\frac{t}{k_1}} \frac{1}{y} g\left(\frac{t}{y}\right) dF(y)dt & \text{if } a \leq x \leq k_2 a \\ \int_{k_2 a}^{x} \int_{\frac{x}{k_2}}^{\frac{x}{k_1}} \frac{1}{y} g\left(\frac{t}{y}\right) dF(y)dt \quad +I_S^M(k_2 a) & \text{if } k_2 a < x < k_1 b \\ \int_{k_1 b}^{x} \int_{\frac{t}{k_2}}^{b} \frac{1}{y} g\left(\frac{t}{y}\right) dF(y)dt \quad +I_S^M(k_1 b) & \text{if } k_1 b \leq x \leq b \end{cases}$$

Thus, $I^M(a) = \int_{k_1 a}^{a} \int_{a}^{\frac{t}{k_1}} \frac{1}{y} g\left(\frac{t}{y}\right) dF(y)dt$ is positive as long as $a > 0$, and $I^M(b) = 1 - \int_{b}^{k_2 b} \int_{\frac{t}{k_2}}^{b} \frac{1}{y} g\left(\frac{t}{y}\right) dF(y)dt$ is strictly less than one as long as $b < \infty$. ∎

Whether imperfect ordinal tool is additive or multiplicative, the imperfect adaptive evaluation is not a distribution function. This is because ordinal imperfections lead to the individual incorrectly perceiving the existence of magnitudes which are outside of the reference set, leading to overvaluation of small magnitudes and undervaluation of large ones.

Example 10 *Suppose the reference distribution F is uniform on $[\mu-\sigma;\mu+\sigma]$, and distribution of multiplicative ordinal discriminability $G(z)$ is also uniform on $[k_1, k_2]$ with $\frac{k_2}{k_1} \leq \frac{\mu+\sigma}{\mu-\sigma}$. Then*

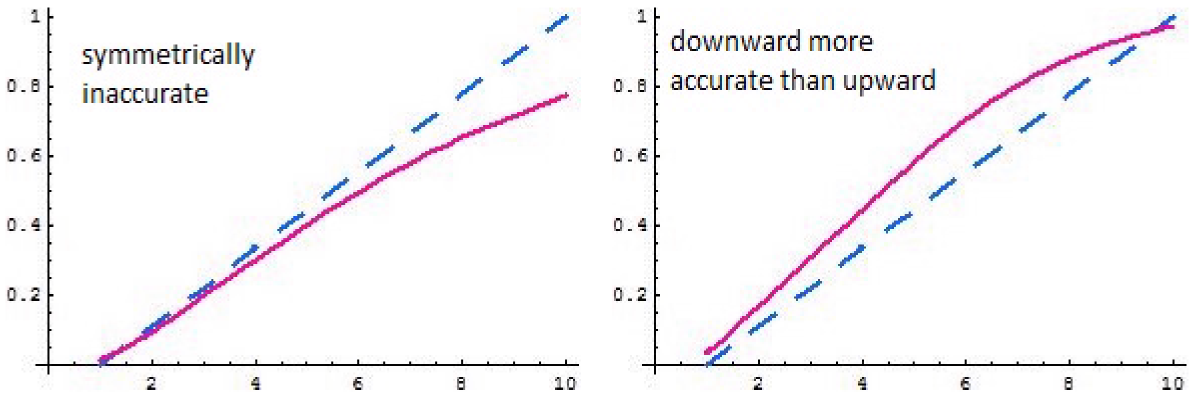
$$I_S^{MU}(x) = \begin{cases} \frac{1}{2\sigma(k_2-k_1)}\left(k_1(\mu-\sigma) - x + x\ln\frac{x}{k_1(\mu-\sigma)}\right) & \text{if } \mu-\sigma \leq x \leq k_2(\mu-\sigma) \\ \frac{\sigma-\mu}{2\sigma} + \frac{x}{2\sigma(k_2-k_1)}\ln\frac{k_2}{k_1} & \text{if } k_2(\mu-\sigma) < x < k_1(\mu+\sigma) \\ 1 - \frac{1}{2\sigma(k_2-k_1)}\left(k_2(\mu+\sigma) - x + x\ln\frac{x}{k_2(\mu+\sigma)}\right) & \text{if } k_1(\mu+\sigma) \leq x \leq \mu+\sigma \end{cases}$$

Thus, in addition to overvaluing small and undervaluing large magnitudes, the evaluation exhibits increasing marginal utility for small magnitudes, decreasing marginal utility for large magnitudes, and constant marginal utility in-between (see Figure 3).

As Figure 3a shows, if an individual faces similar ordinal imperfections for upward and downward comparisons, it is possible that undervaluation may occur for most of the veridical range. In contrast, as Figure 3b shows, if an individual is better at distinguishing magnitudes which are less rather than those which are greater than the target magnitude, most of the veridical magnitudes might be overvalued.

3.6 Economic Applications: Continuous Case

The evaluation procedure could be particularly relevant in cases when evaluations involve interpersonal comparisons, yet cognitive imperfections may prevent individuals from assessing their possessions accurately. Specifically, as the material below demonstrates, when individuals make more inaccurate ordinal comparisons while comparing their possessions to those



a) $x \sim U(1, 10)$, $z \sim U[0.5, 2.0]$ b) $x \sim U(1, 10)$, $z \sim U[0.5, 1.2]$

Figure 3: Relatively to the uniform veridical distribution on $(1, 10)$ (dashed lines) the multiplicatively imperfect evaluation (solid curves) exhibits overvaluation for lower magnitudes and undervaluation for high magnitides.

"above" them, than to those "below" them, their evaluations of own possessions are higher than "true" evaluation, leading to greater satisfaction with their possessions. When individuals have this bias, equality and economic growth may be associated with greater welfare. In general, the intuitively plausible results tend to be associated with ordinal comparison tools which exhibit upward comparison inaccuracies (rather than "fair" judgments or "downward" inaccuracies).

3.6.1 Well-Being and Social Welfare

Consider a private ownership economy with a continuum of individuals, each endowed with an object of magnitude x (e.g. height, intelligence, income, happiness, and so on). The magnitudes of individual endowments are observable and thus constitute the reference set S with the veridical magnitude distribution F. Assume that, being atomistic, all individuals have the same reference set S, have perfect frequency tool F, and the same ordinal tool $\mathcal{D}$.

Individuals evaluate the magnitude of their objects via interpersonal encounters with other individuals in the economy. Each encounter between two individuals possessing magnitude x and y, respectively, thus two outcomes of the ordinal comparison tournament. The result of the ordinal comparison tournament for the individual possessing x is given by $\mathcal{D}(x, y)$, while the corresponding result for the individual possessing y is given by $\mathcal{D}(y, x)$. Let us further aggregate the results of each encounter, and define the *surplus* function $\Psi(x \perp y)$ from an encounter between x and y to be the sum of the results of both ordinal comparison tournaments:

$$\Psi(x \perp y) = \mathcal{D}(x, y) + \mathcal{D}(y, x) \tag{23}$$

Clearly, $0 \leq \Psi(x \perp y) \leq 2$. (For example, if the ordinal tool is perfect, whenever $x < y$,

$\mathcal{D}^P(x,y) = 0$ and $\mathcal{D}^P(y,x) = 1$ (and vice versa), and whenever $x = y$, $\mathcal{D}^P(x,y) = \mathcal{D}^P(y,x) = \frac{1}{2}$, so that for any x, y, $\Psi^P(x \perp y) \equiv 1$.) The surplus function $\Psi(x \perp y)$ plays an important role for the societal welfare. As the following Proposition suggests, the distributions of economically important goods is welfare-neutral if and only if the surplus function Ψ has a constant sum property.

Proposition 16 *Consider a private ownership economy where all individuals have the same reference set S consisting of magnitudes of objects belonging to all other individuals in the economy with veridical (true) magnitude distribution F. Suppose all individuals have the perfect frequency (proportion) tool F, and an identical ordinal comparison tool. Then the utilitarian measure of the total welfare $W = \int_S I(x)dF(x)$ equals to $\frac{C}{2}$ independently of the magnitude distribution $F(x)$ iff the ordinal comparison tool subject to a constant-sum property $\Psi(x \perp y) = \mathcal{D}(x,y) + \mathcal{D}(y,x) = C \in [0,2]$ everywhere except at a countable number of encounters x, y.*

Proof: Rewrite the total welfare as:

$$W = \int_S I_S(x)dF(x) = \int_S \int_S \mathcal{D}(x,y)dF(x)dF(y)$$

Since adaptive evaluation is determined for any x and y, one can also write W as:

$$\begin{aligned} W &= \textstyle\int_S \int_S \mathcal{D}(y,x)dF(y)dF(x) &= \int_S \int_S (\Psi(x \perp y) - \mathcal{D}(x,y))dF(x)dF(y) = \\ & &= \int_S \int_S \Psi(x \perp y)dF(x)dF(y) - W \end{aligned}$$

so that

$$W = \frac{1}{2} \int_S \int_S \Psi(x \perp y)dF(x)dF(y) \tag{24}$$

Clearly, if $\Psi(x \perp y) = C$, then $W = \frac{C}{2}$. Conversely, for W to be independent of F, $\Psi(x \perp y)$ has to be constant. ∎

Corollary 6 *If all individuals in economy have perfect cognitive tools, resulting in Robson-optimal evaluation, then the utilitarian welfare W is independent of the distribution F.*

The economic processes, such as economic growth, and public policies, such as redistributive taxation, tend to change the distribution of individual possessions. As individuals adapt their evaluations of their possessions to the new social environment, the cognitive processes involved in interpersonal comparisons might be important for societal welfare (and thus the success of economic policies). Whenever a pairwise comparison tournament entails opposite outcomes for any two individuals, welfare neutrality of economic processes may arise.[15]

Clearly, with frequency imperfections the welfare neutrality does not hold. With a perfect frequency tool, if the ordinal tool allows for both individuals to "win" the ordinal comparison tournament when the magnitudes x and y are sufficiently similar, it is possible that more equal societal distributions increase welfare. Conversely, equality may decrease welfare if sufficiently similar magnitudes lead to "losses" for both individuals. The next example demonstrates a possibility that different inaccuracies in ordinal comparisons may lead to different social outcomes.

Example 11 *Suppose individuals have additively imperfect evaluations, with the magnitude and ordinal discriminability distributions uniform as in Example 9. Then welfare is given by*

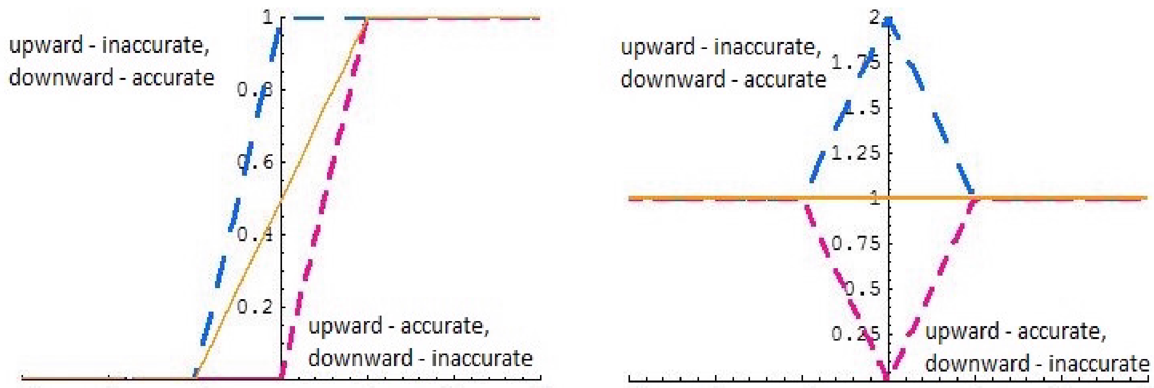
$$W = \frac{2k_1^3 - (k_1 - 2\sigma)^3 + (k_2 - 2\sigma)^3}{24\sigma^2(k_2 - k_1)}$$

Note first that the welfare is independent of average magnitude μ. Yet if ordinal discriminability $z \sim U[-k, k]$ with $k \leq \sigma$, the surplus function $\Psi = 1$ for all x and y, and welfare $W = 0.5$ independently of σ. In contrast, as Figure 4 demonstrates, if ordinal tool is subject to "upward comparison inaccuracies", so that ordinal discriminability $z \sim U[-k, 0]$ with $k \leq 2\sigma$, the surplus function Ψ has maximum at $x = y$ and welfare $W = \frac{1}{2} + \frac{k(6\sigma - k)}{24\sigma^2} > 0.5$. Instead, if ordinal tool is subject to "downward comparison inaccuracies", so that ordinal discriminability $z \sim U[0, k]$ with $k \leq 2\sigma$, the surplus function Ψ has minimum at $x = y$ and welfare $W = \frac{1}{2} - \frac{k(6\sigma - k)}{24\sigma^2} < 0.5$.

The case of multiplicatively imperfect evaluations is more computationally more complex.

Example 12 *Suppose individuals have multiplicatively imperfect evaluations, with the magnitude and ordinal discriminability distributions uniform as in Example 10. The resulting welfare W is a complex quadratic function of the ratio $\frac{\mu}{\sigma}$. Again, there is a relationship between the behavior of the surplus function $\Psi(x \perp y) = \frac{(x/y + y/x) - 2k_1}{k_2 - k_1}$ and welfare W. Specifically, for $k \leq \frac{\mu + \sigma}{\mu - \sigma}$, if ordinal tool is "upward inaccurate", so that ordinal discriminability $z \sim U[1/k, 1]$, the surplus function Ψ has maximum at $x = y$ and welfare W increases with the ratio $\frac{\mu}{\sigma}$. That is, for a given level of inequality σ, economic growth is socially desirable, while for a given level of aggregate income, inequality is not. Instead, if ordinal tool is "downward inaccurate", so that ordinal discriminability $z \sim U[1, k]$, the surplus function Ψ has minimum at $x = y$ and welfare $W =$ decreases with the ratio $\frac{\mu}{\sigma}$.*

[15] Compare this welfare neutrality result to the "happiness paradox" of Easterlin [1974] who pointed out that average self-reported happiness in USA stayed practically unchanged in the post-war period despite real incomes nearly doubled.



a) additive imperfection $z \sim U[k_1, k_2]$ b) surplus function Ψ

Figure 4: When additive ordinal comparisons are "symmetrically inaccurate" (or fair) with $z \sim U[-k, k]$, surplus is $\Psi = 1$ for all x, y (brown lines); when ordinal comparisons are "upward inaccurate" with $z \sim U[-k, 0]$, surplus Ψ is maximized at the point of parity $x = y$ (blue lines), and when when ordinal comparisons are "downward inaccurate" with $z \sim U[0, k]$, surplus Ψ is minimized at the point of parity $x = y$ (red lines).

Multiplicatively imperfect ordinal tools which satisfy constant sum property are rare, but they also result in welfare neutrality.

Example 13 *Suppose veridical income distribution is uniform on* $[\mu - \sigma, \mu + \sigma]$ *and suppose all agents have multiplicatively imperfect ordinal tool on* $[\lambda^{-1}, \lambda]$*, with* $1 < \lambda^2 < \frac{\mu+\sigma}{\mu-\sigma}$ *with ordinal discriminability* $z = \frac{x}{y}$ *being distributed with* $F^{\nu}(z) = 0.5 + 0.75\frac{\ln z}{\ln \lambda} - 0.25\left(\frac{\ln z}{\ln \lambda}\right)^3$*. One can verify that as* $F^{\nu}(z) + F^{\nu}(z^{-1}) = 1$*, the welfare neutrality holds.*

One can speculate that social aspects of interpersonal comparisons alter the nature and shape of ordinal comparison inaccuracies. However, regardless of whether the ordinal comparison is entirely cognitive or has any social component to it, the relationship between welfare and income distribution is determined by the social surplus Ψ function. Specifically, if the surplus is highest at the point of parity $x = y$, equality is likely to be socially desirable, and vice versa.

3.6.2 Relative Skill Judgment

Svenson (1981) observed that individuals tend to misjudge their skills relatively to the others. The phenomenon of individuals overvaluing their own skills relatively to those of the others has since been widely documented. Let us explore whether cognitive imperfections could contribute to such overconfidence.

First, as Propositions 5 and 7 suggest in the case of the perfect frequency tool, the individuals with the lowest possible skill tend to overvalue their skills relatively to the others, and those at the very top of the distribution will tend to undervalue their skills. As Proposition 8 suggests, this tendency maybe further aggravated by frequency imperfections. Second, as Figure 3b suggests, if individuals are have difficulties in making accurate upward comparisons (i.e. in comparing themselves accurately to those whose skills are better than theirs), the overconfidence in relative skill judgment will arise for a significant fraction of the population.

Burks, Carpenter, Goette and Rustichini [2010] further observe that, on average, the least able tend to overvalue their relative skill, and the most able tend to undervalue themselves, with the judgment gap between the judgment of own skill rank and the true skill rank decreasing with the true rank. This phenomenon maybe consistent with possessing cognitive imperfections in their skill evaluations.

Proposition 17 *The judgment gap between assessment of own skill $I(x)$ and true rank $F(x)$ is non-increasing in true skill rank $F(x)$ if and only if the marginal magnitude evaluation $\frac{dI(x)}{dx}$ (which is the density of convolution/scale mixture of pairwise comparison tournament z and comparison variable y) is no higher than the true density $f(x)$ for all skill levels x.*

Proof: For any skill x, let r stand for the true rank in the distribution $F(\cdot)$, or $r = F(x)$. Then, the evaluation of skill x with true rank r will be given by $I(x) = I(F^{-1}(r))$. Consider how the judgment gap $I(x) - F(x) = I(F^{-1}(r)) - r$ changes with rank r:

$$\begin{aligned} \frac{d(I(x) - F(x))}{dF(x)} &= \frac{d(I(F^{-1}(r)) - r)}{dr} = \frac{d \int_S \mathcal{D}(F^{-1}(r), y) dF(y)}{dr} - 1 = \\ &= \frac{1}{f(x)} \int_S \mathcal{D}'(x, y) dF(y) - 1 = \frac{1}{f(x)} \left(\frac{dI(x)}{dx} - f(x) \right) \end{aligned}$$

For additively imperfect ordinal tool, the condition becomes $\frac{dI^A(x)}{dx} = \int_S g(x-y) f(y) dy \leq f(x)$, and for multiplicatively imperfect tool: $\frac{dI(x)^M}{dx} = \int_S \frac{1}{y} g\left(\frac{x}{y}\right) f(y) dy \leq f(x)$. ∎

Corollary 7 *Suppose skill is distributed uniformly on interval $[a, b]$, and that the ordinal comparison tool z is imperfect with density $g(\cdot)$. Then the judgment gap is non-increasing in true rank.*

Proof: $\frac{dI^A(x)}{dx} = \frac{1}{b-a} \int_S g(x-y) dy \leq \frac{1}{b-a}$. Similarly, $\frac{dI^M(x)}{dx} = \frac{1}{b-a} \int_S \frac{1}{y} g\left(\frac{x}{y}\right) dy \leq \frac{1}{b-a}$. ∎

In other words, as long as individuals have difficulties in making upward ordinal comparisons, and cognitive imperfections result in reduced sensitivity to an increase in own skill (as manifested by a less steep marginal evaluation of skill), the imperfect adaptive evaluation is consistent with overconfidence in relative skill judgment.

3.7 Adding Frequency Imperfections

One can accommodate frequency imperfections into continuous model by assuming that, as in Quiggin [1982], imperfect frequency tool can be represented as $w(F)$, with $w' > 0$. Then the adaptive evaluation of magnitude x is

$$\hat{I}_S(x) = \int_S G(Z(x,y))dw(F(y)) \tag{25}$$

That is, the adaptive evaluation is determined by three components: by the context of evaluation F, by the ordinal imperfection G and by the frequency (proportion) weighting $w(F)$, resulting in adaptive evaluation very different from the reference distribution.

Proposition 18 *Suppose an individual's magnitude evaluation is given by (25). With a perfect ordinal comparison tool, his adaptive evaluation of magnitude x is entirely determined by its rank in the weighted magnitude distribution:*

$$I_S^{FP}(x) = w(F(x)) \tag{26}$$

so that $I_S^{FP}(x) \lesseqqgtr I_S^P(x)$ whenever $w(F(x)) \lesseqqgtr F(x)$. Moreover, an adaptive evaluation with imperfect frequency tool is suboptimal in the sense of Robson [2001].

Proof: Using representations (16) and (17), get

$$I_S^{FP}(x) = \int_S \mathcal{D}^P(x,y)dw(F(y)) = \int_S H(x-y)\,dw(F(y)) = \int_S H\left(\frac{x}{y}-1\right)dw(F(y)) = w(F(x)) \quad \blacksquare$$

With perfect ordinal tool, if the frequency imperfections follow the pattern found by Kahneman and Tversky [1979], frequency imperfections tend to boost the adaptive evaluations for relatively small magnitudes and depress evaluations of relatively large magnitudes, further aggravating the distortion caused by the ordinal imperfections.

4 Two-Stage Decision Process: Valuation vs. Choice

With the development of new technology, behavioral scientists became increasingly interested in the processes involved in choice behavior. In their review of the recent findings on the neurobiological mechanism of decision-making in humans and primates, Kable and Glimcher [2009] describe the decision process as a two-stage procedure consisting of a valuation stage (involving ventro-medial prefrontal cortex and striatum) and a choice stage (involving lateral prefrontal and parietal cortices). This leads to a possibility that discrete choice can be decomposed into two distinct processes - the valuation process and choice process.

While economists have been interested in the final, choice, stage, the issue of magnitude evaluation has frequently been overlooked. Yet even for alternatives which differ on a single dimension, like size, one still needs to tell apart the larger object from the smaller object. If one had more-is-better preferences, and there were no constraints on choice, one still needs to know which of all alternatives is the largest (and thus most desirable). In contrast, economists assumed that decision-makers have readily available rankings of all alternatives, and concentrated on the final stage of decision making, choice, in the presence of known ranking. Yet, as this paper suggests, the first, valuation, stage, is an important component of decision-making, understanding of which may clarify choice behavior - even when choice alternatives differ only along one dimension. One thus would expect that when the alternatives have different attributes, the situation would be even more complicated.

Consider Robinson Crusoe on a desert island, and suppose that he faces a choice between a coconut and a banana, which have magnitudes x_c and x_b respectively. Suppose Crusoe is indifferent between coconuts and bananas, but he prefers a "gigantic" banana to a "tiny" coconut (and vice versa) - i.e. he has "more is better" preferences. Thus, before he can make a choice between a coconut and a banana, he needs to be able to evaluate the fruit sizes. Let us suppose that Crusoe's decision making process consists of two stages. In the first, evaluation, stage he evaluates the magnitude x of each fruit, which may then be stored in his memory for the relevant period of time, and in the second, choice stage, conditional on his assessment of fruit sizes, he selects the fruit he finds to be "bigger".

In other words, an individual's decision making process involving a choice between two alternatives can be described by the following algorithm.[16]

Two-Stage Binary Choice Process: *Suppose an individual faces a choice between two alternatives $j = 1, 2$, with magnitudes x_j. Then the process of choice between these two alternatives can be described as follows:*

Stage 1: For each of the choice alternatives $j = 1, 2$, an individual evaluates magnitude x_j using magnitude evaluation functions $I_j(x_j)$, and given these magnitude evaluations $I_j = I_j(x_j)$, forms utilities $U_j = U_j(I_j) = U_j(I_j(x_j))$.

Stage 2: Given the utilities U_j of each object $j = 1, 2$, the individual chooses the object 1 over the object 2 with probability $\Omega(U_1, U_2) \in [0, 1]$ as follows:

$$\Omega(U_1, U_2) = \begin{cases} 0 & \text{whenever} \quad U_1 - U_2 < \nu^- \\ \omega(U_1, U_2) & \text{whenever} \quad \nu^- \leq U_1 - U_2 \leq \nu^+ \\ 1 & \text{whenever} \quad U_1 - U_2 > \nu^+ \end{cases} \tag{27}$$

[16] The two-stage decision-making process presented here is a simplification which ignores the learning process and the dynamic feedback between the past choice results and future choices.

where ν^- and ν^+ are lower and upper thresholds of just noticeable differences of Luce [1956].

The above two-stage procedure allows for different cognitive processes involved at each stage of the decision making (Kable and Glimcher [2009]). Stochastic decision processes and/or perceptual limitations can happen at each stage of the process, leading to random choices.[17] Anderson, de Palma and Thisse [1992] identify two general classes of discrete choice models, both motivated by the psychophysical Weber-Fechner law. Thurstone [1927] pointed out that cognitive limitations can occur at the valuation (measurement) stage, leading to a development of *stochastic utility* models. In contrast, Luce [1956] considered situations where the difference in utilities may not satisfy just noticeable differences subsequently leading to the development of *stochastic choice rule* models. The above two stage procedure unifies these two classes of models, with random utility models being relevant when choice randomness is originated at the first (evaluation) stage because underlying alternatives are not perfectly discriminable in a Thurstonian sense,and random choice rule models being relevant if randomness is originated at the second (choice) stage because "utility is not perfectly discriminable" in a Lucean sense. Traditionally, choice theorists made little distinction between these two stages, by "compounding" the full model $\Omega(U_1(I_1(x_1)), U_2(I_2(x_2)))$ into a "reduced-form" model $\Omega(x_1, x_2)$.

The present paper is agnostic regarding possible processes involved at the second stage of decision making. The second-stage randomness may arise because of the perceptual limitations at the choice stage (similar to Rayo and Becker [2007]), or because of more complex decision process in the brain (as in the drift-diffusion model of Ratcliff [1978] and its extensions - see Fehr and Rangel [2011]). Based on the recent developments in neuroscience, psychology, and economics, Solway and Botvinick [2012] build a computational framework for decision-making, linking the multiple cognitive processes which may potentially occur at different stages of action choice.

Instead, the present paper concentrates on the less explored first, evaluation, stage (and thus could be thought as a member of random utility models), which can be nested into the two-stage procedure presented here, with the random choice rule model potentially implemented at the second, choice, stage. However, to keep the focus on the valuation stage, it is assumed throughout the paper that one has "more is better" preferences, and once the evaluation of alternatives is known, the "larger" objects are chosen. In other words, the model presented here abstracts away from a possibility that an individual may have a "taste" for one *type* of objects over the other (e.g. coconuts vs bananas - captured by general utilities U_c and U_b). Instead, it implicitly employs the following assumption.

Assumption: Suppose $U_j = I_j$ and $\nu^- = \nu^+ = 0$.

[17] There is a possibility that perceptual limitations may lead to "incompleteness of preferences", which can happen at either of the stages of the decision making.

This assumption allows one to concentrate on the features of behavior originated at the valuation stage. Such approach underscores the potential mental effort spent evaluating choice alternatives - especially unfamiliar ones. Once the evaluation is done, the choice is trivial. On the other hand, the results of past evaluations could be stored in memory, so that the valuation stage for familiar choice options may be less resource-demanding. The focus on the valuation stage could be particularly relevant in situations where evaluations are of particular importance, for example, for judgment of one's well-being or one's abilities.

5 Related Literature

This paper is related to a large and diverse body of the existing literature in economics, psychology, and neurobiological sciences only a small portion of which is surveyed below.

The model presented here follows the new evolutionary framework for decision-making as an alternative to neoclassical utility theory. Robson [2001] advanced an argument in support of a utility function which adapts to the environment.[18] Rather than endowing living organisms with all necessary information, Nature provides the "tools" that enable one to extract information from one's environment and experience (Robson [2001, 2002], Samuelson [2004], Samuelson and Swinkels [2006]), which might have been further adapted to deal with more evolutionarily recent tasks (Cosmides and Tooby [1994], Gigerenzer [1998]). The quest for evolutionary basis of decision-making primitives has been expanded by Robson and Samuelson [2010] to decision and experienced utilities, while Rayo and Becker [2007] further suggest that Nature may endow humans with happiness as a context-dependent measurement tool which allow one to choose among the alternatives. Furthermore, Herold and Netzer [2011] suggested that the non-linear probability weighting function evolved as an evolutionary second-best to complement an adaptive S-shaped value function.

The evaluation mechanism proposed here assumes that the two primitive cognitive tools are used to extract information from one's environment, rather than to rely on the "absolute" physical properties. Robson [2001] proposed that the optimal hedonimeter is an empirical distribution of the environmental stimuli. Suppose one faces a choice between two alternatives, which are non-negative numbers independently drawn from a continuous distribution F. In terms of hedonic utility, the organism can only observe whether an alternative is above or below each of the N threshold values. That is, hedonic values (e.g. "High", "Low") are assigned following a series of ordinal comparisons to thresholds. When both draws fall within the same region, they are of the same hedonic value. Thus incorrect choices may occur. When both draws fall within the same region, as determined by adjacent threshold values, they are assigned the same hedonic value, despite being distinct. Robson shows that, with no complexity costs, the overall probability of error is minimized when the threshold

[18] See a survey on hedonic adaptation by Frederick and Loewenstein [1999].

values c_i satisfy $F(c_i) = \frac{i}{N+1}$ (e.g. if $N = 99$, each threshold level is a percentile). That is, the optimal hedonic utility function is the distribution function F. This result was further examined by Netzer [2009] who further pointed out that the hedonic discriminability is particularly important for stimuli which are particularly environmentally relevant - that is, the utility function should be more sensitive for the stimuli which are particularly frequent.

Independently, visual scientists Yang and Purves [2004] argued that for the survival of a living organism the empirical significance of stimuli is more important than their physical properties. As the same visual aspect of an environment (such as luminance) in different contexts results in different perception of that aspect, they proposed that the visual system does not record the physical aspects of stimuli, but instead their statistical relevance as represented by their relative rank in the distribution of stimuli.

The present paper is closely related the decision-by-sampling theory by psychologists Stewart, Chater and Brown [2006], who, referring to the formal model of Kornienko [2004], suggested that using a series of binary, ordinal comparisons, an object's subjective value is constructed to be its rank in a sample drawn from one's memory. They show that this procedure may account for a number of observed regularities involving money, time, and probability, including concave utility functions, losses looming larger than gains, hyperbolic time discounting, and the overestimation of small and underestimation of large probabilities.

Ranking alternatives by pairwise comparisons was explored by Rubinstein [1980], who considered ranking participants in a tournament by counting the number of times a given player beats another player in a tournament (with "ties" not allowed). Landau [1951] used trichotomous outcomes of pairwise tournaments to describe societies' dominance structures.

The present paper shares the frequentist procedurial focus on subjective experience with the case-based decision framework pioneered by Gilboa and Schmeidler [1995], which utilizes a database of past experiences to construct decision weights akin probability weights of expected utility. This framework was extended in Gilboa and Schmeidler [2003] to show that one can use empirical frequency distributions to rank an alternative to be "at least as likely" as another.

The present paper adds to a number of theoretical models explicitly or implicitly employing the isomorphism of a cumulative density function and a utility function. One of the the pioneers of context-dependent evaluation, psychologist Parducci [1963, 1965] observed that the shape of subjects' magnitude evaluations followed the corresponding veridical distributions, albeit with less pronounced curvature. To explain such empirical relationship between magnitude evaluations the and context, he proposed his range-frequency theory whereby the magnitude evaluation is a weighted sum of a veridical rank and a rank in a uniform distribution with the same support. Simultaneously and independently from Parducci, the isomorphism of a utility function and a cumulative density function was noted by Van Praag [1968] who developed a formal model with the money utility being isomorphic to a normal distribution. This idea was later extended by Kapteyn [1985] (and references therein). who suggested that money utility is a cumulative density of an income distribution.

In addition to the above rank-based models, there exist a number of reference-dependent models where alternatives are evaluated, at least in part, against a reference set, or its specific elements. Single-reference point theories developed by psychologists include the adaptation-level theory of Helson [1948] and the prospect theory of Kahneman and Tversky [1979].

With a notable exception of the endogenous reference point theory of Koszegi and Rabin [2006], reference-dependent models developed by economists have been primarily concerned with social comparisons. Pioneered by Duesenberry [1949], "keeping up with the Joneses" models involve comparisons to the average action by the others. The observation that individuals care about their relative position in a social hierarchy led Frank [1985] to develop a model where ordinal rank with respect to the other individuals is a component of one's utility. Harbaugh and Kornienko [2000] consider how the individual reference sample from the common environmental information may affect individual valuations.

Lazear and Rosen [1981], followed by Green and Stokey [1983], suggested that comparisons relatively to peers provide informational improvements. This was further advanced by Samuelson [2004] who highlighted the importance of environmental persistence in an evolutionary context. Rayo and Becker [2007] develop an evolutionary-inspired model with perceptual limitations which allows for hedonic effects of relative success arising due to the increased power of the happiness measurement by using average of peer performance. Wolpert [2010] explores the socially constructed benchmarks using both single point (average) and whole population (rank) models.

The empirical evidence suggests that nearly everything that enters one's mental "in-box" is evaluated against a reference set, or its specific elements. There is plenty of evidence that people compare their possessions relatively to what others have (see, for example, Frank [1985], Clark and Oswald [1996], Solnick and Hemenway (1998), and Neumark and Postlewaite [1998]). Nor are moral judgements absolute - as students were found to be appalled by the idea of poisoning a neighbour's dog, yet judged it as a petty crime when poisoning the neighbour herself was on the list (Parducci [1968]). Even evaluation of painful experience is not absolute as it varies with different sequences of pain (Redelmeier and Kahneman [1996]). And so are the judgements of size, weight, or numerousness (Parducci [1963, 1965]), as well as visual perception (Yang and Purves [2004]).

Olivola and Sagara [2009] argue against context-independent utility functions, and show that empirical distribution of death tolls affects experimental subjects' decisions involving human fatalities in hypothetical situations, and moreover, consistent with cross-country differences in environmental contexts, subjects' choices vary across countries. Stewart [2009] found that evaluations by experimental subjects depend on the relative rank of magnitudes in the reference sets. Wood, Brown and Maltby [2011] found that gratitude is affected by the magnitude distribution of help others provide. Stewart, Reimers, and Harris [2011] further argue that since utility, probability weighting and temporal discounting functions are context-dependent, they cannot be stable primitives of economic decision making.

There are a few studies which provide an empirical support for the models based on the isomorphism of evaluation/utility and an empirical rank. Based on the experimental and survey data, Clark, Masclet and Villeval [2006] suggest that one's rank in the income distribution is a strong determinant of effort, while Brown, Gardner, Oswald and Qian [2008] found that one's rank in salary distribution determines job satisfaction and can predict job separation. Using Gallup World Poll data, Barrington-Leigh [2012] reports that, comparing to a specification involving log income, pure ordinal rank is just as successful in explaining the data on life satisfaction, and within country pure rank model provides a better fit.

Furthermore, the brain studies suggest that the anticipated reward magnitudes are coded in relative, and not absolute, terms (Seymour and McClure [2008] and references therein). Mullett and Tunney [2013] recently found that in a human fMRI experiment using monetary values, activations in ventral striatum and thalamus are "locally" context dependent, while activations in ventro-medial prefrontal cortex and the anterior cingulate cortex are consistent with encoding "global" ordinal rank across two different "local" contexts. Their experimental design involved valuation of monetary values independently of choice. This is in contrast to earlier studies on primate neurons in situations where choice and valuation could potentially be confounding, the activity in the orbitofrontal cortex was reported to be invariant to changes of menu (Padoa-Schioppa and Assad [2008]), yet to be adapting to the range of alternatives (Padoa-Schioppa [2009]).

6 Conclusions

This paper provides a conceptual framework which combines the insights from economics, behavioral, evolutionary and cognitive sciences into one parsimonious mathematical model. It advances a hypothesis that a minimal set of cognitive tools is sufficient to extract information from one's environment efficiently, and that utility can be thought as an adaptive measuring tape for choice alternatives.

Undoubtedly, the model presented here has limitations. First, it concerns the magnitude evaluation stage of decision process, rather than with the final choice. The neurobiological processes at the choice stage of decision process is an area of active research (see Fehr and Rangel [2011]), which can be incorporated into the two-stage decision model independently of the processes at the valuation stage. Second, the model describes the evaluation of one-dimensional objects, while individuals are likely to use additional tools to evaluate multidimensional objects and bundles. One possible model for the evaluation of multidimensional bundles (such as bundles consisting of apples and oranges) could employ copulas $U(x_a, x_b) = C(F_a(x_a), F_b(x_b))$, with complements and substitutes modeled, respectively, by positive and negative dependence. Third, the present paper leaves frequency imperfections largely unexplored. It also ignores possible interactions between the frequency (rarity) of particular magnitudes and cognitive/memory imperfections. Brown and Matthews [2011] extend decision-by-sampling model to allow for memory distortions which interact with the reference distributions. Fourth, it remains to be determined whether there are additional cognitive tools involved in processing of uncertainty, as Andreoni and Sprenger [2010] document a qualitative difference between certain and uncertain utility. Fifth, the paper does not explore the evolutionary basis of cognitive imperfections. As Herold and Netzer [2011] suggest, certain cognitive imperfections may evolve to counteract the effects of others. Sixth, it is still unclear how agents evaluate novel items. Examples of early trade between Westerners and Amerindians (e.g. Miller and Hamell [1986]) suggest a possibility that individuals may assess novel objects against the existing systems of values, the process which could possibly be modeled using the similarity functions of Gilboa and Schmeidler [1995]. Seventh, the strategic nature of interpersonal comparisons has been entirely left out. Undoubtedly, asymmetric information could play an important role in evaluation of economically relevant goods, thus adding an extra complication to welfare assessments.

Yet, despite its shortcomings, the present paper highlights the importance of recent advances in cognitive and brain research for understanding of economic decision making.

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