A Cognitive Basis for Context-Dependent Utility

Tatiana Kornienko *†
School of Economics
University of Edinburgh
Edinburgh, EH8 9JT, UK

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Abstract

This paper provides a new way to understand the concept of a utility function. By looking into the cognitive processes of information acquisition, I demonstrate how a utility function can be derived from deeper first principles. Crucially, assuming that humans possess only a minimal set of cognitive tools - ordinal comparison and proportion (or frequency-of-occurrence) estimation, I show that the resulting utility of an object is isomorphic to the perceived rank of its magnitude within a reference set. By construction, the resultant utility is adaptive, and thus is evolutionarily advantageous. With perfect cognitive tools, this adaptive utility minimizes mistakes in binary choices. Cognitive imperfections may lead to suboptimal choices, as small objects tend to be overvalued and large objects tend to be undervalued. The proposed parsimonious model has features that are strongly consistent with empirical observation. This new analytical approach offers a bridge between modern economics, psychology, and neuroscience.

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†tatiana.kornienko@ed.ac.uk, http://homepages.ed.ac.uk/tkornie2/

1 Introduction

How big is an apple? How high is a salary? How intense is an experience? How long is a piece of string? How can one tell?

Recently a number of researchers advanced a new evolutionary framework for decision-making as an alternative to neoclassical utility theory. Robson [2001] suggested that a mistake-minimizing utility function involves adaptation to the environmental context in which the decision is made,[1] and such *optimal* adaptive utility will coincide with the cumulative density function (rank) of the magnitude distribution of alternatives.[2] Furthermore, rather than endowing humans with all necessary information, Nature gives humans "tools" that allow to extract information from one's environment and experience (Robson [2001, 2002], Samuelson [2004], Samuelson and Swinkels [2006]).[3] While individuals may have hard-wired "tools" to evaluate quantities of food, drink and shelter in order to survive, it is questionable that individuals are born with abilities to evaluate evolutionary novel goods - such as exotic fruits, rare wines, designer clothes, the newest technological gadgets, or money (Robson [2001, 2002]). Thus, the ancient tools that our hunter-gatherer ancestors had long ago must have been adapted to deal with more evolutionarily recent tasks (Cosmides and Tooby [1994]).

In contrast to the existing literature which assumes a built-in adaptive utility, I explore *how* an adaptive utility can be acquired. I show that it can be derived from deeper first principles by focusing on the *process* and *tools* by which an individual could *evaluate* a magnitude of an item. I identify two well-documented domain-independent cognitive tools that Nature has endowed humans, and, using a parsimonious mathematical model, show that an *adaptive evaluation* of a magnitude can be built entirely upon *ordinal comparisons* which, by means of a *frequency* (proportion) tool, are keyed into a universal scale. This is done by calculating how frequently a given object "wins" a pairwise ordinal "tournament" against all other objects in the reference set.[4] As the result, an adaptive evaluation of a magnitude equals to its *perceived* relative rank (cumulative density) within the magnitude distribution of the reference objects.[5]

This adaptive evaluation rationalizes well-behaved "relatively more is better" preferences defined on a reference set. However, as Tversky and Kahneman [1991] pointed out, "preference anomalies" are a common consequence of a change in a "reference state".

[1] See a survey on hedonic adaption by Frederick and Loewenstein [1999].

[2] This result was further examined by Netzer [2009], who, in addition, proposed an alternative fitness-maximizing utility function which also depends on the environment.

[3] Rayo and Becker [2007] further suggest that Nature may endow humans with happiness as a context-dependent measurement tool which allow one to choose among the alternatives.

[4] Rubinstein [1980] considered ranking participants in a tournament by counting the number of times a given player beats another player in a tournament. Landau [1951] used trichotomous outcomes of pairwise tournaments to describe societies' dominance structures.

[5] For an earlier work on isomorphism of a utility function and a cumulative density function see Van Praag [1968], Kapteyn [1985], and Castagnoli and LiCalzi [1996].

Indeed, once the reference set changes, so does the adaptive evaluation, with a possibility of non-reflexivity, non-transitivity, and preference reversals when evaluations are compared across distinct reference states. Moreover, a given magnitude is evaluated higher if the reference set is positively rather than negatively skewed.[6]

In a closely related paper, psychologists Stewart, Chater and Brown [2006] found that the evaluations by experimental subjects indeed depend on the relative rank of magnitudes in the comparison sets. They suggested that individuals evaluate an object by its rank in a sample drawn from one's memory, using a series of binary, ordinal comparisons, and that this "decision by sampling" procedure may account for a number of observed regularities involving money, time, and probability, including concave utility functions, losses looming larger than gains, hyperbolic time discounting, and the overestimation of small probabilities and the underestimation of large probabilities (see also Stewart [2009]).[7] Moreover, Stewart, Reimers, and Harris [2010] further argued that since utility, probability weighting and temporal discounting functions are context-dependent, they cannot be stable primitives of economic decision making.

Among the pioneers of context-dependent evaluation, a psychologist Parducci [1963, 1965] developed his range-frequency theory to explain qualitative, but not quantitative, similarities between subjects' magnitude evaluations and veridical reference distributions, with the magnitude evaluation being "flatter" than the veridical distribution.[8] I suggest that such systematic but distinct evaluation may instead arise because people's cognitive tools tend to be imperfect. For example, most people would notice the difference between having one dollar bill and two dollar bills in one's wallet, but would hardly notice the difference between having 67 and 68 dollar bills, and they also may process frequencies, or proportions, imperfectly. In the presence of such cognitive imperfections the *perceived* rank of a magnitude is only partially affected by the veridical rank, thus exhibiting the qualitative similarities with the veridical distribution, while also depending on the form and parameters of the ordinal and frequency imperfections.

I employ a convolution technique whereby an adaptive evaluation can be expressed as either a sum or a product of two independent random variables, namely a reference magnitude and a pairwise tournament success. I show that, if the two cognitive tools are perfect, the adaptive magnitude evaluation is equal to the *veridical* (true) rank in the distribution of reference magnitudes. In other words, as long as the two primitive cognitive tools are perfect, the adaptive evaluation is *optimal* in a sense of Robson [2001]. I also show that if each individual in the economy has the same reference set,

[6] See Parducci [1963, 1965], Seidl, Traub, and Morone [2005], and Stewart, Chater and Brown [2006] for experimental evidence.

[7] Brown, Gardner, Oswald, and Qian (2008) applied this approach to explain job satisfaction, while Olivola and Sagara [2009] found that empirical distribution of death tolls affects experimental subjects' decisions involving human fatalities in hypothetical situations. Wood, Brown and Maltby [2011] find that gratitude is affected by the magnitude distribution of help others provide.

[8] The range-frequency theory of Parducci [1963, 1965] involves the magnitude evaluation being the weighted sum of a veridical rank and a rank in a uniform distribution with the same support.

has a perfect frequency tool, and uses the same ordinal comparison tool subject to a zero-sum condition, a utilitarian measure of welfare is independent of the distribution of economic goods. Thus, human adaptive evaluation could lead to the famous paradox of happiness (Easterlin [1974], [1995]), whereby the average self-reported happiness is almost identical across time and nations despite individual happiness is increasing in one's own income and aggregate incomes vary widely.

The advantage of the convolution model is that it can accommodate imperfections in cognitive tools. I show that an imperfect evaluation differs from the veridical evaluation in the implied risk attitudes, as smaller magnitudes tend to be overestimated, and larger tend to be underestimated, with such distortion being more severe when both ordinal and frequency (proportion) tools are imperfect.[9] The imperfect evaluation is thus suboptimal in a binary choice and precludes the welfare neutrality of income distribution. Even when the veridical distribution is uniform (implying risk neutrality), I demonstrate that a specific imperfect ordinal tool leads to more risk loving at the lower end and more risk aversion at the upper end, and can lead to risk aversion for most part of the range.

The mathematical model presented here has been influenced by the relative lottery comparisons of Castagnoli and LiCalzi [1996].[10] The perfect ordinal tool presented here is isomorphic to the optimal happiness function of Rayo and Becker [2007], who model perceptual imperfections involved when the expected happiness is compared across different choice options, rather than built into the two cognitive tools considered here.

There is plenty of evidence that people compare their possessions relatively to what others have (see, for example, Frank [1985], Clark and Oswald [1996], Solnick and Hemenway (1998), Neumark and Postlewaite [1998]). Nor are moral judgements absolute - as students were found to be appalled by the idea of poisoning a neighbour's dog, yet judged it as a petty crime when poisoning the neighbour herself was on the list (Parducci [1968]). Even evaluation of painful experience is not absolute as it varies with different sequences of pain (Redelmeier and Kahneman [1996]). And so are the judgements of size, weight, or numerousness (Parducci [1963, 1965]), as well as visual perception (Yang and Purves [2004]). Furthermore, the brain studies suggest that the values are coded in relative, and not absolute, terms (Seymour and McClure [2008]).

Thus, nearly everything that enters the human "in-box" is evaluated against a reference set, or its specific elements. This observation led economists and psychologists to develop a number of reference-dependent theories, including the "keeping up with the Joneses" theory of Duesenberry [1949], the adaptation-level theory of Helson [1948], the range-frequency theory of Parducci [1963, 1965], neo-cardinal theories of Van Praag [1968] and Kapteyn [1985 and references therein], the social rank concerns of Frank [1985], the prospect theory of Kahneman and Tversky [1979], the endogenous reference

[9] The imperfect evaluation is consistent with the overconfidence patterns found by Burks, Carpenter, Goette and Rustichini [2010].

[10] See also Kacelnik and Abreu [1998] for a similar model of risky choice with perceptual imperfections.

points of Koszegi and Rabin [2006], and the decision by sampling of Stewart, Chater and Brown [2006].

Here, it is shown that only two primitive tools are sufficient to "measure" a magnitude against a reference set. One such primitive tool - ordinal comparisons - enables one to tell whether a magnitude is larger, smaller, or equal to another magnitude. Brannon [2002] found that 9-11 months old human infants are able to discriminate between small arrays of objects, while Feigenson, Carey and Spelke [2002] suggested that human infants develop abilities to discriminate continuous variables (such as areas, sizes, densities) even earlier, while Xu [2003] reported that large numerosities are discriminated differently from the small ones, and Walsh [2003] suggested that time, space and numbers involve the similar neural mechanisms. The other primitive tool is frequency-of-occurrence. Cognitive and evolutionary psychologists report that humans update frequencies easily and constantly (Hintzman and Stern [1978]), as well as automatically and accurately (Hasher and Zacks [1979, 1984]). Since human children and even animals are equipped with a mental system for counting (Gallistel and Gelman [1992]), natural frequencies can be stored in a numerical format (Jonides and Jones [1992]).[11]

The issue of magnitude evaluation and measurement imperfections dates back to the 19th century, and is closely associated with the law of diminishing marginal utility. According to the Weber's law of just noticeable differences, or Weber-Fechner-Stevens law (see, for example, Laming [1973]), humans fail to notice the difference between two magnitudes if it is less than a certain threshold, and the minimum noticeable difference is proportional to the stimulus level. This phenomenon was recently found on the brain level, as Nieder and Miller [2003] found that the representation of numerosities is "compressed" in a monkey brain, prompting Dehaene [2003] to advance an argument in support of a logarithmic mental number line to represent the Weber-Fechner law.[12]

The research on frequency (proportion) imperfections is still in its infancy. As Kahneman and Tversky [1979] pointed out, humans tend to overweight small probabilities and underweight large ones (see also Camerer and Ho [1994], Prelec [1998], and Gonzalez and Wu [1999]), and this probability distortion may happen at the encoding level in the brain, as Hsu, Krajbich, Zhao and Camerer [2009] found that the neural activity in the striatum is nonlinear in stimulus probabilities.[13] Psychologists have accumulated further evidence that humans tend to evaluate proportions (such as probabilities, proportions of graphical elements, numerosities, an so on) in a non-linear fashion, involving accurate evaluations at the "intermediate" proportions, overestimation of low and underestimate of large proportions (see Spence [1990] and Hollands and Dyre [2000]).

[11] The literature on frequency processing prompted a revision of human abilities to deal with uncertainty - see Gigerenzer and Hoffrage [1995] and Cosmides and Tooby [1996] among others.

[12] Dehaene, Izard, Spelke, and Pica [2008] found that education has a "linearizing" effect on the mental number scale, as in Amazonian indigenous cultures the mental number line is found to be logarithmic, while in the Western societies it was found to be closer to the linear.

[13] Herold and Netzer [2011] suggested that the non-linear probability weighting function evolved as an evolutionary second-best to complement an adaptive S-shaped value function.

By providing a parsimonious mathematical model which incorporates cognitive imperfections, the present paper may build a bridge between the economic models of decision making and experimental research on human psychophysics and neuroscience.

2 The Basic Idea

Before I present the model in Section 4, I first present the basic idea behind the model.

Consider Robinson Crusoe on a desert island. There is a single coconut tree on the island, which bears coconuts in a variety of sizes, and Crusoe can see all coconuts at once. One day, a coconut falls off the tree. How would Crusoe evaluate this coconut? Clearly, if the coconut is the smallest one, it is the least valuable. If instead it is the largest one, it is the most valuable. If it is neither smallest nor largest, its evaluation is in-between. In particular, the value of a given coconut depends on the proportion of the coconuts which are smaller, equal, and larger than the given coconut.

More formally, suppose that there is a set X of magnitudes of N observable objects of the same type (e.g. sizes of physical goods such as coconuts, houses, cars, incomes, and so on, or magnitudes of a characteristic such as height, beauty, intelligence, etc.). An individual is endowed with two simple cognitive tools on the set X. First, he is able to make ordinal comparisons of any two magnitudes in the set X, i.e. for any two magnitudes x and $\tilde{x}$ in the set X, he is able to tell that either $x > \tilde{x}$, or $x < \tilde{x}$, or $x = \tilde{x}$. Second, he able to assess the proportions (or frequency of occurrence) of smaller, equal, or larger items. As the following algorithm shows, these two tools are sufficient to construct an evaluation of a magnitude x relatively to a reference set X.

Adaptive Evaluation Algorithm *Suppose an individual faces a comparison set X and is endowed with ordinal comparison and frequency (proportion) processing tools. Then the adaptive evaluation of a magnitude x can be constructed as follows:*

Step 1 *Using the ordinal comparison tool, compare the magnitude x to every other magnitude x' in the reference set X - that is, observe whether $x > x'$, or $x < x'$, or $x = x'$.*

Step 2 *Using the frequency (proportion) processing tool, estimate how often $x > x'$, $x < x'$, and $x = x'$, i.e. the values $\frac{N_{\tilde{x}<x}}{N}$, $\frac{N_{\tilde{x}>x}}{N}$, and $\frac{N_{\tilde{x}=x}}{N}$, respectively, where $N_{\tilde{x}<x}+N_{\tilde{x}=x}+N_{\tilde{x}>x} = N$.*

Step 3 *Evaluate the magnitude x relatively to the reference set X as follows:*

$$I_X(x) = 1 \cdot \frac{N_{\tilde{x}<x}}{N} + C \cdot \frac{N_{\tilde{x}=x}}{N} + 0 \cdot \frac{N_{\tilde{x}>x}}{N} \qquad (1)$$

where $C \in [0, 1]$ is a constant representing the value of a "tie".

The value of a "tie" $C \in [0,1]$ assigned when magnitudes x and $\tilde{x}$ are perceived to be similar. It can be interpreted as a tendency for the directional comparison. When $C = 0.5$, upward and downward comparisons are symmetric. When $C < 0.5$, an individual tends to compare downwards. In particular, if $C = 0$, he makes downward comparisons only, and the adaptive evaluation $I_X(x)$ is equal to the perceived probability of meeting a smaller magnitude, or ordinarily "outperforming" a randomly drawn magnitude from the reference set X. Similarly, when $C > 0.5$, an individual tends to compare upwards, and with $C = 1$ he makes upward comparisons only, leading to $I_X(x)$ being the probability of meeting no larger magnitude from the reference set X. As it will be shown later, when the ordinal comparison tool is imperfect, a tendency to perceive objects similar may play an important role for an adaptive evaluation.

Evidently, the adaptive evaluation $I_X(x)$ on the reference set X is increasing in the magnitude x. Let us extend the notion of *reference structure* introduced by Tversky and Kahneman [1991] to define a preference relation indexed to the entire reference set X. Assume that the reference structure $\succeq_X$, which one can call "relatively more is better" preferences, is complete and transitive. Then, given the reference set X, the reference structure $\succeq_X$ is rationalizable by the adaptive evaluation $I_X(\cdot)$ on the reference set X.

The above adaptive evaluation procedure allows for updating, so as an evaluation $I_X(x)$ of a magnitude x changes with a change in the reference set X. As Tversky and Kahneman [1991] showed that "reference shift", or change in a reference state, often results in preference reversals. Thus, similarly to other reference-dependent models, when the reference set X changes, the adaptive evaluation $I_X(\cdot)$ may exhibit apparent preference anomalies (such as non-reflexivity, non-transitivity and preference reversals) when the choices are compared across different reference states.

As the adaptive evaluation procedure is context-dependent, Robinson Crusoe's evaluation is affected by the size distribution of coconuts on his island, and will change if he were on another island. For example, Crusoe would be less happier with a 3 inch coconut on an island A where a typical coconut is 5 inch radius rather than on another island B where a typical coconut is 2 inch radius - simply because coconuts smaller than 3 inch are more frequent on island B.[14] Thus it is possible to find a banana such that Crusoe will be happy to trade the 3-inch coconut for this banana on island A, but on island B he would be happy to trade this banana back for the 3-inch coconut - an apparent preference reversal.[15]

To evaluate a coconut, Robinson Crusoe compares it against other coconuts present - whether these other coconuts are on a tree, on a store display, or in hands of other

[14] While such behavior could be explained by "aspirations", it is possible that aspirations are affective consequence of the sentiment-free adaptive evaluation process.

[15] If Robinson Crusoe is destined to have the same coconut (or height, intelligence, and so on) for life, he would choose the island where his coconut wins the pairwise tournaments most frequently. Yet, the present model is silent regarding his choice over islands when other coconuts (income, consumption) could also be available.

people. From the cognitive point of view, Mrs. Brown evaluates her house the same way as Crusoe evaluates a coconut, - using two primitive cognitive tools over a reference set consisting of other house, - yet these reference houses tend to belong to other people. In a private ownership economy most (if not all) ordinal comparisons are necessarily done interpersonally, particularly for individual characteristics such as height, intelligence, beauty, as well as economically relevant variables such as consumption, income, and wealth. These interpersonal comparisons have important economic consequences as a change in Mr. Jones' possessions changes how Mrs. Brown evaluates what she has, appearing as a desire to "keep up with the Joneses".[16]

3 Introducing Imperfections

The advantage of the model of Section 4, is that it allows to accommodate imperfections in the cognitive tools into adaptive evaluation. To see how cognitive imperfections may have an effect on adaptive evaluation, consider, for example, two sets, $A = \{10, 11, 12, 12, 98, 98, 98, 98, 98, 99, 100\}$ and $B = \{10, 11, 12, 12, 12, 12, 12, 98, 98, 99, 100\}$, and consider the adaptive evaluations of magnitudes 11 and 99 against the two sets. Since each magnitude has the same ordinal rank in both sets, their adaptive evaluations will also be the same, or $I_A(11) = I_B(11) = \frac{1}{10}$ and $I_A(99) = I_B(99) = \frac{9}{10}$.

Let us now suppose that the ordinal comparison tool is imperfect in the sense that any two magnitudes *perceived* to be similar will "tie" in a pairwise ordinal comparison. As psychologists discovered in XIX century, humans may not notice the difference between two relatively similar magnitudes, the phenomenon known as Weber-Fechner-Stevens law (see Laming [1973]). Thus, an individual may perceive the two above-mentioned sets as follows: $A' = \{11, 11, 11, 11, 99, 99, 99, 99, 99, 99, 99\}$ and $B' = \{11, 11, 11, 11, 11, 11, 11, 99, 99, 99, 99\}$. Now, following the same procedure with value of the tie $C = 0.5$, the adaptive evaluations of magnitudes of 11 and 99 are

$$\begin{aligned} I_{A'}(11) &= \tfrac{(3)(0.5)}{10} &> I_A(11) \\ I_{A'}(99) &= \tfrac{4+(6)(0.5)}{10} &< I_A(99) \\ I_{B'}(11) &= \tfrac{(5)(0.5)}{10} &> I_B(11) \\ I_{B'}(99) &= \tfrac{6+(4)(0.5)}{10} &< I_B(99) \end{aligned}$$

Here, the imperfect ordinal comparison tool leads to overvaluing of small magnitudes and undervaluing of large ones. Also the composition of the reference set becomes even more important when ordinal imperfections are present - as, relatively to its true rank, magnitude 11 is overvalued less in set A', where smaller magnitudes are less frequent

[16]While an adaptive evaluation is sentiment-free, in private ownership economies it may lead to a concern with social status discussed by Veblen [1899], Frank [1995], and many others.

than in set B'. Conversely, magnitude 99 is undervalued less in set B', where larger magnitudes are less frequent than in set A'.

Now, *instead*, suppose that the frequency tool is imperfect. As Kahneman and Tversky [1979] pointed out individuals may overweigh small probabilities and underweigh large ones, while Hollands and Dyre [2000] found even a more complex phenomenon for proportion judgments over a variety of stimuli (including continuous sensory modalities such as fullness of the glass, etc.). Furthermore, as the example below demonstrates, overestimation of cumulative frequency for small magnitudes and underestimation for large ones may be an artefact *counting* error, or imperfections in numerosity judgments. As Dehaene, Izard, Spelke, and Pica [2008] found that subjects tend to lose count, and undercount large numerosities (particularly for sets of dots and sequences of tones), and that subjects from indigenous societies tend to overcount small numerosities.

Thus, one can speculate that, in large comparison sets, the total number of elements is undercounted, while the number of smaller items is overcounted for small magnitudes (because there is only a small number of smaller items), and is undercounted for large magnitudes (because there is a large number of smaller items). This process can be rationalized by some elements with small magnitudes (e.g. the element with magnitude 10) perceived as having a "twin", and some elements with large magnitudes (e.g. two elements with magnitudes 98) perceived to be absent. For example, sets A and B could be perceived as $A'' = \{10, 10, 11, 12, 12, 98, 98, 98, 99, 100\}$ and $B'' = \{10, 10, 11, 12, 12, 12, 12, 12, 99, 100\}$. In this case, the adaptive evaluations of each magnitude are the same in both sets, however they are different from the true ranks, with magnitude 11 being overvalued and 99 being undervalued:

$$\begin{aligned} I_{A''}(11) &= \tfrac{2}{9} > I_A(99) \\ I_{B''}(11) &= \tfrac{2}{9} > I_A(99) \\ I_{A''}(99) &= \tfrac{8}{9} < I_A(99) \\ I_{B''}(99) &= \tfrac{8}{9} < I_A(99) \end{aligned}$$

Now, let us combine the two imperfections, so that the sets A and B are perceived as $A''' = \{11, 11, 11, 11, 11, 99, 99, 99, 99, 99\}$ and $B''' = \{11, 11, 11, 11, 11, 11, 11, 11, 99, 99\}$, and the adaptive evaluations are:

$$\begin{aligned} I_{A'''}(11) &= \tfrac{(4)(0.5)}{9} > I_A(11) \\ I_{A'''}(99) &= \tfrac{5+(4)(0.5)}{9} < I_A(99) \\ I_{B'''}(11) &= \tfrac{(6)(0.5)}{9} > I_B(11) \\ I_{B'''}(99) &= \tfrac{7+(1)(0.5)}{9} < I_B(99) \end{aligned}$$

Thus, when both imperfections are present, small magnitudes are overvalued and large magnitudes are undervalued more dramatically than when only one tool is imperfect.

As the above examples demonstrate, the cognitive imperfections and the reference set composition interact in a complex way, which, for expositional simplicity, will be further explored for continuous magnitude distributions.

4 The Basic Model

Let the reference set be $X = [x_{min}, x_{max}] \subseteq R^+$, and let the reference magnitudes be distributed with a continuously differentiable *perceived* distribution function $\hat{F}(\cdot)$ on X, with $\hat{F}'(\cdot) > 0$ on X. It is easy to see that (cumulative density) function $\hat{F}(x)$ is a *frequency tool* as $F(x) = Pr(\tilde{x} \leq x)$, or equal to the frequency with which $\tilde{x}$ is less or equal than x. To define the *ordinal comparison tool*, let an outcome of pairwise comparison of a fixed magnitude x with any magnitude $\tilde{x}$ be represented by the *pairwise comparison function* $\mathcal{D}(x, \tilde{x}) : X \to [0, 1]$, which is non-decreasing in x and non-increasing in $\tilde{x}$, and is discontinuous at a countable number of points. Then, for any magnitude $x \in X$, define the adaptive evaluation $I_X(x)$ of x against reference set X as an expected outcome of pairwise comparisons:

$$I_X(x) = \int_X \mathcal{D}(x, \tilde{x}) d\hat{F}(\tilde{x}) \tag{2}$$

so that $I_X(x)$ is the expected perceived outcome of pairwise comparison tournament of x against every element $\tilde{x}$ in comparison set X, or the perceived rank of x in X. That is, the adaptive evaluation is parsimoniously constructed with two basic cognitive tools, with the ordinal comparison tool being the integrand (or kernel of integration), and the frequency (or proportion) tool being the variable of integration.[17] The next result follows from the assumptions on the pairwise comparison function $\mathcal{D}$.

Proposition 1 *Suppose an individual is endowed with a frequency (proportion) tool* $\hat{F}(\cdot)$ *and a pairwise comparison tool* $\mathcal{D}(x, \tilde{x})$*, which is non-decreasing in* x *and non-increasing in* $\tilde{x}$*. Then*

(i) $I_X(x)$ *is a non-decreasing continuous function on set* X*.*

(ii) if the reference set A *is perceived to stochastically dominate set* B*, i.e.* $\hat{F}_A \succeq_{FOSD} \hat{F}_B$*, then* $I_B(x) \geq I_A(x)$ *for any* $x \in X$*.*

Proof: (i) Since $\mathcal{D}$ is increasing in x and is discontinuous at a finitely many points, $I_X(x)$ is non-decreasing and continuous.

(ii) By definition of first order stochastic dominance (FOSD), for any non-decreasing function $U(t)$, $\int_A U(t) d\hat{F}_A(t) \geq \int_B U(t) d\hat{F}_B(t)$. But since $\mathcal{D}$ is non-increasing in $\tilde{x}$, we have that

$$I_A(x) = \int_A \mathcal{D}(x, \tilde{x}) d\hat{F}_A(\tilde{x}) \leq \int_B \mathcal{D}(x, \tilde{x}) d\hat{F}_B(\tilde{x}) = I_B(x) \quad \blacksquare$$

[17] The basic model is closely related to Castagnoli and LiCalzi [1996] (expected) probability model.

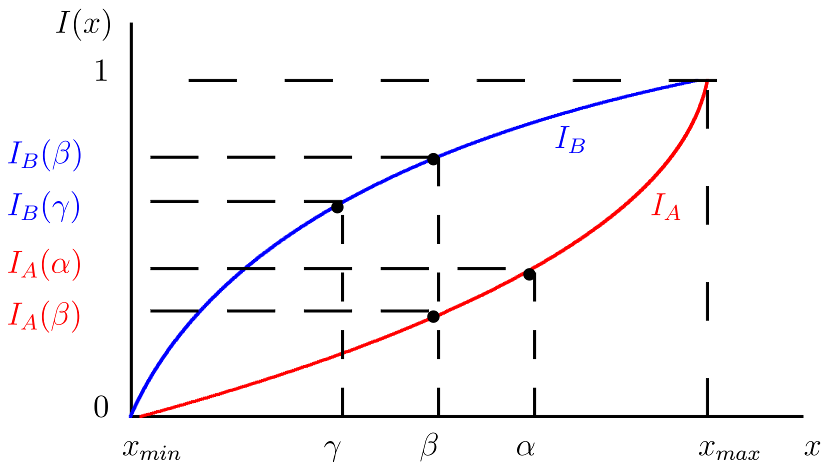


Figure 1: Violations of reflexivity and transitivity across different reference sets for $\hat{F}_A \succeq_{FOSD} \hat{F}_B$.

Thus, the adaptive evaluation $I_X(x)$ can rationalize continuous more is better preferences on X. Further, once the context changes, so does the adaptive evaluation. As Figure 1 shows, both reflexivity and transitivity may be violated. That is, one can find some $\beta \in A \cap B$, such that, first, $I_A(\beta) \neq I_B(\beta)$, and, second, it is possible to find $\alpha \in A$ and $\gamma \in B$ such that $I_A(\alpha) > I_A(\beta)$ and $I_B(\beta) > I_B(\gamma)$, but $I_A(\alpha) < I_B(\gamma)$. This has an important consequence. Suppose there exist an "outside" object with exogenously determined valuation $I_Y(\delta)$ where $(A \cup B) \cap Y = \emptyset$ such that $I_A(\beta) < I_Y(\delta) < I_B(\beta)$. Then, an individual would trade β for δ when reference set is A, but once reference set changes to B, he would trade δ back for β - an apparent preference reversal.

This formulation allows to integrate cognitive imperfections into an adaptive evaluation. In particular, suppose the ordinal comparison tool is noisy, so that the result of a pairwise tournament is a random variable z distributed as $G(z)$. Then, for particular forms of noisy ordinal tool, one can model the adaptive evaluation as a convolution of an ordinal comparison tool and a frequency (proportion) tool, highlighting the role of cognitive imperfections for adaptive evaluation.

Proposition 2 *Let the result of the pairwise tournament be represented by a random variable z distributed as $G(z)$ and the magnitude distribution be perceived as $\hat{F}(\cdot)$. Suppose that the evaluation of a magnitude x is a function of two independent random variables, a reference magnitude $\tilde{x}$ and a tournament success variable z. Then the adaptive evaluation $I_X(x)$ is a convolution of $\tilde{x}$ and z:*

$$I_X(x) = \mathcal{D} * \hat{F} = \int G(z) d\hat{F}(\tilde{x}) = \begin{cases} \int_X G\left(\frac{x}{\tilde{x}}\right) d\hat{F}(\tilde{x}) & \text{if} \quad x = z\,\tilde{x} \\ \int_X G\left(x - \tilde{x}\right) d\hat{F}(\tilde{x}) & \text{if} \quad x = z + \tilde{x} \end{cases} \tag{3}$$

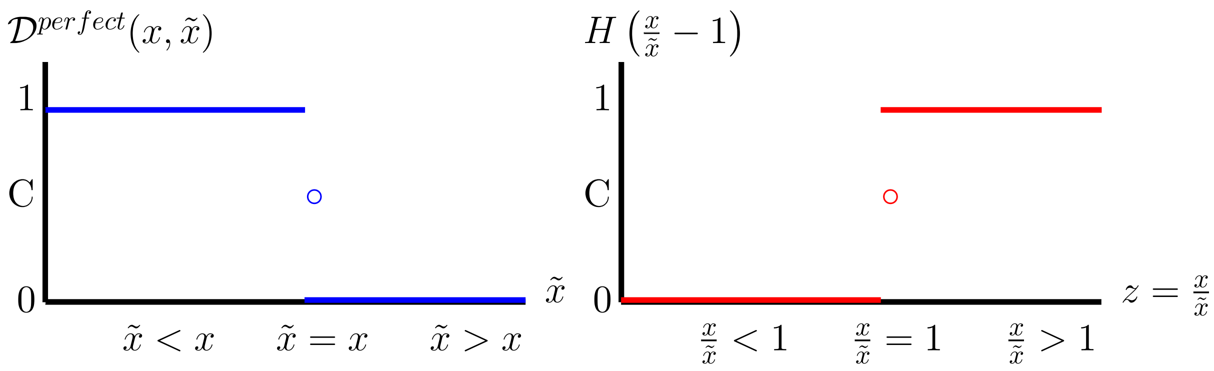


Figure 2: Perfect pairwise comparison function $\mathcal{D}^{perfect}(x,\tilde{x})$ and the distribution of the corresponding tournament success variable $z=\frac{x}{\tilde{x}}$, where x is magnitude of interest, $\tilde{x}$ - reference magnitude.

5 Adaptive Evaluation with Perfect Cognitive Tools

Adaptive evaluation with perfect cognitive tools provides a useful benchmark for subsequent analysis. Suppose an individual is endowed with a *perfect* frequency (or proportion) tool, so that he correctly perceives the veridical distribution of magnitudes $F(\cdot)$. Suppose further he is endowed with a *perfect* ordinal comparison tool. That is, he can tell a bigger from a smaller magnitude even if the magnitudes are only slightly different. In this case, an outcome of a pairwise comparison can be written as:[18]

$$\mathcal{D}^{perfect}(x,\tilde{x}) = \begin{cases} 1 & \text{whenever} \quad x>\tilde{x} \\ C & \text{whenever} \quad x=\tilde{x} \\ 0 & \text{whenever} \quad x<\tilde{x} \end{cases} \tag{4}$$

where $C\in[0,1]$ (see Figure 2, left panel). This perfect (noiseless) ordinal comparison tool can be represented by a degenerate tournament success random variable z,[19] using either a multiplicatively noisy or an additively noisy formulation. If $x=z\,\tilde{x}$, then $z=\frac{x}{\tilde{x}}$ is degenerate at 1, with $G(z)=H(z-1)=H\left(\frac{x}{\tilde{x}}-1\right)$ (see Figure 2, right panel). If, instead, $x=z+\tilde{x}$, then $z=x-\tilde{x}$ is degenerate at 0, with $G(z)=H(z)=H(x-\tilde{x})$. Since $H\left(\frac{x}{\tilde{x}}-1\right)=H(x-\tilde{x})$, the two formulations represent the same random variable.

Proposition 3 *Suppose an individual is endowed with a perfect ordinal comparison tool (4) and with a perfect frequency (proportion) tool $F(\cdot)$. Then his adaptive evaluation of magnitude x is given by the cumulative density function of the veridical magnitude distribution $F(x)$:*

$$I_X^{perfect}(x)=F(x) \tag{5}$$

Moreover, the perfect adaptive evaluation is optimal in a sense of Robson [2001].

[18] Compare this perfect ordinal tool to the optimal happiness function of Rayo and Becker [2007].

[19] A random variable z which is degenerate at a is represented as having a domain $(-\infty,\infty)$, with a Dirak delta density function $\delta(x-a)$, and a cumulative distribution being a Heaviside (step) function $H(z-a)$, with $H(z-a)=0$ for $z<a$, $H(z-a)=1$ for $z>a$, and $H(a)\in[0,1]$.

Proof: Use the multiplicative representation of $\mathcal{D}^{perfect}$, i.e. let $z = \frac{x}{\tilde{x}}$ with $G(z) = H\left(\frac{x}{\tilde{x}} - 1\right)$. Then the adaptive evaluation is a product convolution:

$$I_X^{perfect}(x) = \int_X \mathcal{D}^{perfect}(x, \tilde{x}) dF(\tilde{x}) = \int_X H\left(\frac{x}{\tilde{x}} - 1\right) dF(\tilde{x}) = F(x)$$

The optimality result follows from Robson [2001] and Netzer [2009]. ∎

That is, if both ordinal comparison and frequency (proportion) tools are perfectly accurate, an individual's evaluation $I_X(x)$ of magnitude x can be described by the veridical cumulative density function $F(x)$, and is equal to the frequency of a magnitude x ordinarily "outperforming" other elements, or to its veridical rank $F(x)$ on the reference set X.[20] Moreover, as it was shown by Robson [2001] and Netzer [2009], if the adaptive utility is $F(x)$, the probability of mistake in a binary choice is minimized.

When cognitive tools are perfect, the shape of the magnitude distribution determines attitudes toward risk. For any veridical distribution $F(x)$, the Arrow-Pratt measure of absolute risk aversion is $r(x) = -\frac{U''(x)}{U'(x)} = -\frac{F''(x)}{F'(x)}$ and of relative risk aversion $xr(x)$. Thus, if the veridical magnitude distribution is of the power function form, individual's adaptive evaluation is consistent with a constant relative risk aversion (CRRA) utility function; and if the veridical distribution is exponential, the evaluation exhibits constant absolute risk aversion (CARA).[21]

Consider now a private ownership economy with a continuum of individuals, each endowed with an object of magnitude x (e.g. height, intelligence, income, happiness, and so on). Suppose that these magnitudes of individual endowments are observable and thus constitute the reference set X with the veridical magnitude distribution $F(\cdot)$. Assume that, being atomistic, all individuals have the same reference set X, and have identical cognitive tools, with ordinal comparison tool subject to a "zero-sum property":

$$\mathcal{D}(x, \tilde{x}) + \mathcal{D}(\tilde{x}, x) = 1 \tag{6}$$

That is, a pairwise comparison tournament has opposite outcomes for any two individuals, with a favorable outcome for one individual implying an unfavorable outcome for the other. (The perfect comparison tool $\mathcal{D}^{perfect}(x, \tilde{x})$ with $C = 0.5$ is among those which satisfy the zero-sum property.) The zero-sum property leads to an interesting result.

Proposition 4 *Consider a private ownership economy where all individuals have the same reference set X consisting of magnitudes of objects belonging to all other individuals in the economy. Suppose the individuals have identical ordinal comparison tool subject to the zero-sum property (6), as well as the perfect frequency (proportion) tool $F(\cdot)$. Then*

[20] This isomorphism of a utility function and a cumulative density function was first noticed by Van Praag [1968], and further explored by Kapteyn [1985 and references therein].

[21] See Castagnoli and Li Calzi [1996] for these relationships between distribution functions and the shape of utility function.

the utilitarian measure of the total welfare $W = \int_X I(x)dF(x)$ *is independent of the magnitude distribution and equals to 0.5.*

Proof: Rewrite the total welfare as:

$$W = \int_X I_X(x)dF(x) = \int_X \int_X \mathcal{D}(x, \tilde{x})dF(x)dF(\tilde{x})$$

Since adaptive evaluation is determined for any x and $\tilde{x}$, and $\mathcal{D}(x, \tilde{x})$ satisfies the zero-sum property (6), one can write W as:

$$W = \int_X I_X(\tilde{x})dF(\tilde{x}) = \int_X \int_X \mathcal{D}(\tilde{x}, x)dF(\tilde{x})dF(x) = \int_X \int_X (1 - \mathcal{D}(x, \tilde{x}))dF(x)dF(\tilde{x})$$

Equating the two expressions for W, get

$$W = \int_X \int_X \mathcal{D}(x, \tilde{x})dF(x)dF(\tilde{x}) = 0.5 \quad \blacksquare \tag{7}$$

That is, the economic processes, such as economic growth, and public policies, such as redistributive taxation, tend to change the distribution of individual possessions. But as individuals adapt their evaluations of their changed possessions to the new social environment, some cognitive tools may lead to welfare neutrality of economic processes.[22]

6 Imperfect Ordinal Comparisons

To understand the role of imperfections for the adaptive evaluation procedure, let us first explore the role of imperfections in the ordinal comparison tool only. Suppose that an individual is endowed with a *perfect* frequency tool, or that the perceived distribution $\hat{F}(x)$ is exactly equal to the veridical (true) distribution $F(x)$. (This assumption will be relaxed later.) Once the ordinal imperfections are taken into account, an adaptive evaluation will not only be affected by the veridical magnitude distribution $F(\cdot)$, but also on the ordinal noise $G(\cdot)$, resulting in an adaptive evaluation being functionally distinct from the veridical distribution. Given that imperfect tools lead to evaluation functions distinct from the veridical distribution function, it is clear that such evaluations are suboptimal in a sense of Robson [2001], as well as preclude the welfare neutrality of income distribution (7).

[22] Compare this welfare neutrality result to the "happiness paradox" of Easterlin [1974], [1995] who pointed out that average self-reported happiness in USA stayed practically unchanged in the post-war period despite real incomes increased nearly doubled during this time.

6.1 Multiplicatively Noisy Ordinal Comparison Tool

Let ordinal comparison tool be subject to a multiplicative noise z, i.e. $x = z\,\tilde{z}$, where z is distributed with $G(z)$. In this case the result of ordinal comparison tournament is:

$$\mathcal{D}^{multi-noise}(x, \tilde{x}) = \begin{cases} 1 & \text{whenever } \frac{x}{\tilde{x}} > z \\ C & \text{whenever } \frac{x}{\tilde{x}} = z \\ 0 & \text{whenever } \frac{x}{\tilde{x}} < z \end{cases} \tag{8}$$

Thus, whenever $z > 1$, a comparison magnitude $\tilde{x}$ "looms large", with the ordinal tournament assessment of magnitude x being biased downwards. And vice versa, whenever $z < 1$, a comparison magnitude $\tilde{x}$ "looms small", and the evaluation of x is boosted upwards. The resulting adaptive evaluation is a product convolution of the veridical distribution and the noise distribution.

Proposition 5 *Suppose an individual is endowed with a perfect frequency (proportion) tool $F(\cdot)$, and a multiplicatively noisy ordinal comparison tool (8) with noise z distributed with $G(z)$. Then, his adaptive evaluation of a magnitude x is:*

$$I^{multi}(x) = \int_X \int_Z H\left(\frac{x}{\tilde{x}} - z\right) dF(\tilde{x})dG(z) = \int_X G\left(\frac{x}{\tilde{x}}\right) dF(\tilde{x}) = \int_Z F\left(\frac{x}{z}\right) dG(z) \tag{9}$$

Proof: Note that

$$\begin{aligned} I^{multi}(x) &= \int_X G\left(\frac{x}{\tilde{x}}\right) dF(\tilde{x}) = \int_X \int_Z H\left(\frac{x}{\tilde{x}} - z\right) dF(\tilde{x})dG(z) = \\ &= \int_X \int_Z H\left(\frac{x}{z} - \tilde{x}\right) dF(\tilde{x})dG(z) = \int_Z F\left(\frac{x}{z}\right) dG(z) \ \blacksquare \end{aligned}$$

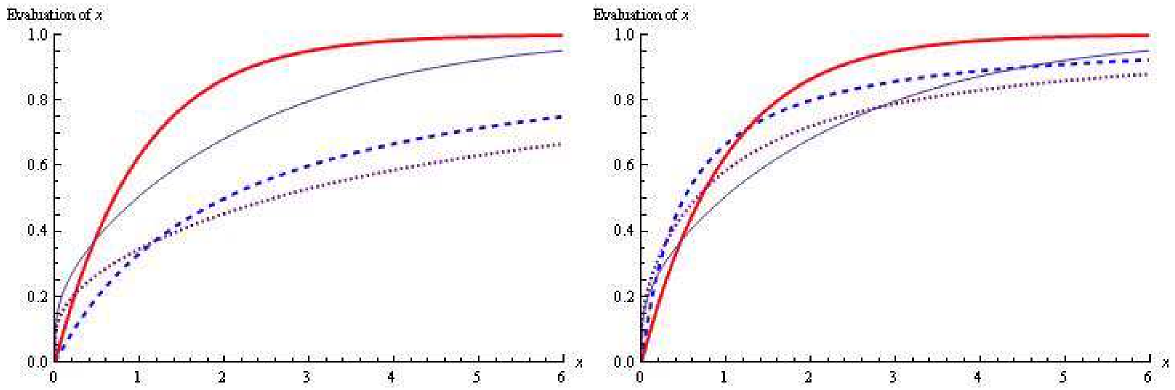
The following result demonstrates that a multiplicatively noisy comparison tool results in an adaptive evaluation functionally distinct from the veridical distribution of magnitudes, altering risk preferences (see Figure 3).

Proposition 6 *Suppose the veridical magnitude distribution is exponential, i.e. $F(x) = 1 - e^{-\theta x}, x \in [0, \infty), \theta > 0$, implying constant absolute risk aversion (CARA). Suppose an individual is endowed with a perfect frequency (proportion) tool, and an ordinal comparison tool subject to multiplicative noise with the reciprocal $z^{-1} = \tau$ being Gamma-distributed, i.e. $\tilde{g}(\tau) = \frac{\beta^\alpha}{\Gamma(\alpha)}\tau^{\alpha-1}e^{-\beta\tau}, \tau \in [0, \infty), \alpha > 0, \beta > 0$ and $E(\tau) = \frac{\alpha}{\beta}$. Then,*

(i) his adaptive evaluation of magnitude x is given by Lomax (Pareto II) distribution:

$$I_X^{Lomax}(x) = 1 - \left(\frac{\beta}{\theta x + \beta}\right)^\alpha \tag{10}$$

(ii) his absolute risk aversion is hyperbolic (HARA) with $r(x) = \frac{\theta(\alpha+1)}{\theta x+\beta}$;



a) "Depressing" ordinal noise ($\alpha \leq \beta$) b) "Boosting" ordinal noise ($\alpha > \beta$)

Figure 3: The exponential veridical distribution $F(x)$ (thick solid curves) differs from the ordinarily noisy evaluation $I_X^{Lomax}(x)$ (10) (dashed curves). Both are further distorted when Prelec [1998] frequency (proportion) imperfection $w(F(x)) = \exp[-(-lnF(x))^\delta], \delta \in (0,1)$ is included, with twisted veridical distribution (thin solid curves) being distinct from twisted noisy evaluation (dotted curves).

(iii) if ordinal noise is "depressing" with $\alpha \leq \beta$, then $I_X(x) < F(x)$ for all $x > 0$, and if ordinal noise is "boosting" with $\alpha > \beta$ then there exists $x^* > \frac{\alpha-\beta}{\theta}$ such that $I_X(x) > F(x)$ for $x < x^*$ and $I_X(x) < F(x)$ for $x > x^*$.

Proof: (i) The result follows from Johnson, Kotz and Balakrishnan (1994, p.574):

$$I_X^{Lomax}(x) = \int_\tau F(\tau x) g(\tau) d\tau = \int_0^\infty \left(1 - e^{-\theta x \tau}\right) \frac{\beta^\alpha}{\Gamma(\alpha)} \tau^{\alpha-1} e^{-\beta\tau} d\tau = 1 - \left(\frac{\beta}{\theta x + \beta}\right)^\alpha$$

(ii) The veridical distribution $F(x)$ implies constant absolute risk aversion with $r(x) = \theta$, and linearly increasing relative risk aversion $xr(x) = \theta x$. The multiplicatively noisy evaluation $I_X^{Lomax}(x)$ implies decreasing and convex absolute risk aversion with $r(x) = \frac{\theta(\alpha+1)}{\theta x+\beta}$, and increasing and concave relative risk aversion with $xr(x) = (\alpha+1)\left(1 - \frac{\beta}{\theta x+\beta}\right)$, approaching in the limit to the constant relative risk aversion equal to $\alpha + 1$.

(iii) To see that F and I_X cross at most once, first compare their slopes at $x = 0$:

$$\lim_{x\to 0}(F'(x) - I'_X(x)) = \lim_{x\to 0}(\theta e^{-\theta x} - \theta\alpha\beta^\alpha(\theta x + \beta)^{-\alpha-1}) = \frac{\theta}{\beta}(\beta - \alpha)$$

Thus, if $\alpha < \beta$ (i.e. the the mean value of reciprocal of noise z is greater than 1, so that the ordinal noise z tends to boost magnitude x in ordinal comparisons), then $F(x)$ is steeper than $I_X(x)$ at the origin, and vice versa. Also note that $F(x)$ approaches 1 faster than $I_X(x)$, and thus $F(x) > I_X(x)$ as $x \to \infty$. Thus, if $\alpha < \beta$, then $F(x)$ and $I_X(x)$ either do not cross, or cross twice. If $\alpha > \beta$ then $I_X(x)$ crosses $F(x)$ exactly once, and from above. If a point of crossing x^* exists then $F(x^*) = I_X(x^*)$, and $F'(x^*) \gtrless I'_X(x^*)$

iff $x^* \gtrless \frac{\alpha-\beta}{\theta}$. If $F(x)$ and $I_X(x)$ do cross, then, given that $F(0) = I_X(0) = 0$, there exists a point $\hat{x}$ such that $F'(\hat{x}) = I'_X(\hat{x})$, and that $F(\hat{x}) \gtrless I_X(\hat{x})$ iff $\hat{x} \gtrless \frac{\alpha-\beta}{\theta}$. Thus, if $\alpha \leq \beta$ then $I_X(x) < F(x)$ for all $x > 0$, and if $\alpha > \beta$ then $I_X(x) > F(x)$ for $x < x^*$ and $I_X(x) < F(x)$ for $x > x^*$ where $x^* > \frac{\alpha-\beta}{\theta} > \hat{x}$. Finally, if $\alpha = \beta$, then $x = 0$ is the only point where $F' = I'_X$ and $F = I_X$. ∎

For example, if wealth is exponentially distributed, the veridical evaluation implies that even the very rich are reluctant to take gambles proportional to their wealth. In contrast, the multiplicatively noisy (Lomax) adaptive evaluation implies risk aversion decreases as wealth increases, and the rich are more prone to take gambles which are proportional to their wealth. Moreover, when ordinal noise tends to "blow up" comparison values, every magnitude is undervalued relatively to its veridical evaluation (see Figure 3a), but when the ordinal noise tends to "boost" the magnitude of interest, small magnitudes are overvalued and medium and large ones are undervalued (see Figure 3b).

6.2 Additively Noisy Ordinal Comparison Tool

Let the ordinal comparison tool be subject to an additive noise z, i.e. $x = z + \tilde{z}$, where z is distributed with $G(z)$. Here, the result of ordinal comparison tournament is:

$$\mathcal{D}^{add-noise}(x, \tilde{x}) = \begin{cases} 1 & \text{whenever} \quad x - \tilde{x} > z \\ C & \text{whenever} \quad x - \tilde{x} = z \\ 0 & \text{whenever} \quad x - \tilde{x} < z \end{cases} \tag{11}$$

Thus, whenever $z > 0$, a comparison magnitude $\tilde{x}$ "looms large", with the ordinal tournament assessment of magnitude x being biased downwards. And vice versa, whenever $z < 0$, a comparison magnitude $\tilde{x}$ "looms small", and the evaluation of x is boosted upwards. The resulting adaptive evaluation is a sum convolution of the veridical distribution and the noise distribution.

Proposition 7 *Suppose an individual is endowed with a perfect frequency (proportion) tool $F(\cdot)$, and an additively noisy ordinal comparison tool (11) with noise z distributed with $G(\cdot)$. Then, his adaptive evaluation of a magnitude $x$is:*

$$I_X^{add}(x) = \int_X \int_Z H\left((x - \tilde{x}) - z\right) dF(\tilde{x}) dG(z) = \int_X G(x - \tilde{x}) dF(\tilde{x}) = \int_Z F(x - z) dG(z) \tag{12}$$

Proof: Note that

$$\begin{aligned} I^{add}(x) &= \int_X G\left(x - \tilde{x}\right) dF(\tilde{x}) = \int_X \int_Z H\left((x - \tilde{x}) - z\right) dF(\tilde{x}) dG(z) = \\ &= \int_X \int_Z H\left((x - z) - \tilde{x}\right) dF(\tilde{x}) dG(z) = \int_Z F\left(x - z\right) dG(z) \ ∎ \end{aligned}$$

That is, the additively noisy adaptive evaluation is convolution sum of reference magnitudes and noise. For example, suppose the reference magnitudes $\tilde{x}$ are normally distributed with mean μ_x and variance σ_x^2 and the noise z is also normally distributed with mean $\mu_z = 0$ and variance σ_z^2. Then the additively noisy adaptive evaluation of magnitude x will be given by the rank in the convolution distribution which is normal with mean $\mu_x + \mu_z = \mu_x$ and variance $\sigma_x^2 + \sigma_z^2$. Given that the convolution distribution will have "fatter" tails, the relatively low magnitudes are overvalued and the relatively high magnitudes are undervalued.

6.3 Weberian Ordinal Comparison Tool

The psychophysical Weber-Fechner-Stevens law states that humans tend not notice the difference between two magnitudes if it is less than a certain threshold, and in particular, the minimum amount by which stimulus intensity must be changed in order to produce a noticeable variation increases with the stimulus level.[23] Here, I will assume that the comparison tool is Weberian in a sense of Rubinstein [1988], i.e. two distinct magnitudes are judged to be similar (equal) if the ratio of their magnitudes is neither large enough nor small enough. In addition, I assume that within the "similarity domain" (interval of uncertainty) the ordinal comparisons are random.

Suppose that an individual can tell apart two distinct magnitudes x and $\tilde{x}$ with certainty only if the ratio of magnitudes is outside of an interval $[\phi, \psi]$, with $0 < \phi < 1 < \psi$, but within this interval an individual can only tell them apart with some probability. Thus, one can write the ordinal comparison tool as follows:

$$\mathcal{D}^{Weber}(x, \tilde{x}) = \begin{cases} 1 & \text{whenever } \frac{x}{\tilde{x}} > \psi \\ G\left(\frac{x}{\tilde{x}}\right) & \text{whenever } \phi \leq \frac{x}{\tilde{x}} \leq \psi \\ 0 & \text{whenever } \frac{x}{\tilde{x}} < \phi \end{cases} \tag{13}$$

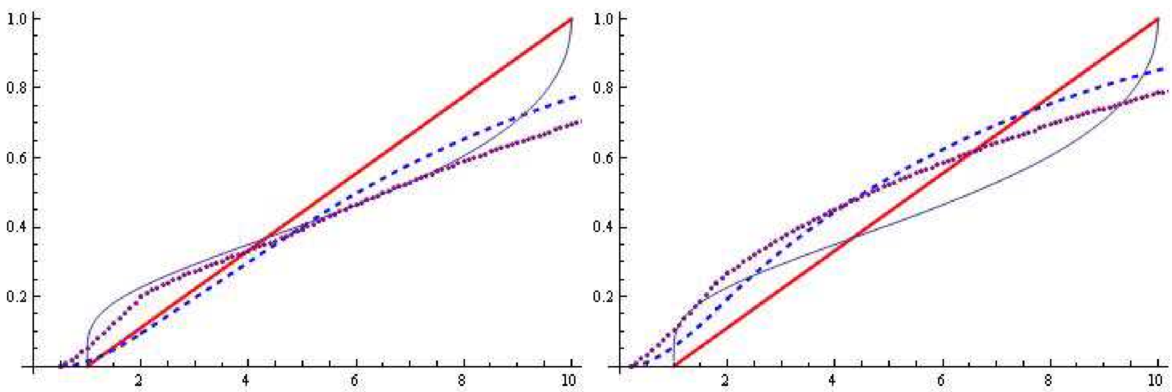
where $G(z)$ is a distribution of z on $[\phi, \psi]$. The next result follows from Glen, Leemis and Drew [2004].

Proposition 8 *Suppose an individual is endowed with a perfect frequency (proportion) tool $F(\cdot)$, and a Weberian ordinal comparison tool (13) with noise z distributed with $G(\cdot)$ on $[\phi, \psi]$. Then, his marginal evaluation $\frac{dI_X^{Weber}(x)}{dx}$ is given by:*

$$\frac{dI_X^{Weber}(x)}{dx} = \begin{cases} \int_{x_{min}}^{\frac{x}{\phi}} g\left(\frac{x}{\tilde{x}}\right) \frac{1}{\tilde{x}} dF(\tilde{x}) & \text{if } \phi x_{min} < x < \gamma_1 \\ \int_{\Delta_1}^{\Delta_2} g\left(\frac{x}{\tilde{x}}\right) \frac{1}{\tilde{x}} dF(\tilde{x}) & \text{if } \gamma_1 < x < \gamma_2 \\ \int_{\frac{x}{\psi}}^{x_{max}} g\left(\frac{x}{\tilde{x}}\right) \frac{1}{\tilde{x}} dF(\tilde{x}) & \text{if } \gamma_2 < x < \psi x_{max} \end{cases}$$

where $\gamma_1 = \psi x_{min}, \gamma_2 = \phi x_{max}, \Delta_1 = \frac{x}{\psi}, \Delta_2 = \frac{x}{\phi}$ if $\psi x_{min} < \phi x_{max}$; $\gamma_1 = \phi x_{max}, \gamma_2 = \psi x_{min}, \Delta_1 = x_{min}, \Delta_2 = x_{max}$ if $\psi x_{min} > \phi x_{max}$; and $\gamma_1 = \gamma_2 = \psi x_{min} = \phi x_{max}$ if $\psi x_{min} = \phi x_{max}$. The adaptive evaluation I_X^{Weber} is given by the integral of the above.

[23] See, for example, Laming [1973] for formal treatments of this phenomena.



a) Weberian noise $z \sim U[0.5, 2]$ b) Weberian noise $z \sim U[0.2, 1.8]$

Figure 4: On the veridical range $[1, 10]$, the uniform veridical distribution $F(x)$ (thick solid curves) differs from the Weberian evaluation $I_X^{Weber}(x)$ (13) (dashed curves). With frequency (proportion) imperfection $w(F(x)) = \exp[-(-lnF(x))^{0.5}]$, twisted veridical distribution (thin solid curves) is distinct from twisted noisy evaluation (dotted curves).

Corollary 1 *Suppose the veridical distribution $F(\cdot)$ is uniform on $[x_{min}, x_{max}]$, and that an individual is endowed with a perfect frequency (proportion) tool $F(\cdot)$, and a Weberian ordinal comparison tool (14) with uniform noise z on $[\phi, \psi]$. When $\psi x_{min} < \phi x_{max}$,*

$$\frac{dI_X^{U-Weber}(x)}{dx} = \begin{cases} \frac{1}{\psi-\phi}\frac{1}{x_{max}-x_{min}}\left(\ln(x) - \ln(\phi x_{min})\right) & \text{if } \phi x_{min} < x < \psi x_{min} \\ \frac{1}{\psi-\phi}\frac{1}{x_{max}-x_{min}}\left(\ln(\psi) - \ln(\phi)\right) & \text{if } \psi x_{min} < x < \phi x_{max} \\ \frac{1}{\psi-\phi}\frac{1}{x_{max}-x_{min}}\left(\ln(\psi x_{max}) - \ln(x)\right) & \text{if } \phi x_{max} < x < \psi x_{max} \end{cases}$$

When $\psi x_{min} > \phi x_{max}$ the result is quantitatively similar:

$$\frac{dI_X^{U-Weber}(x)}{dx} = \begin{cases} \frac{1}{\psi-\phi}\frac{1}{x_{max}-x_{min}}\left(\ln(x) - \ln(\phi x_{min})\right) & \text{if } \phi x_{min} < x < \phi x_{max} \\ \frac{1}{\psi-\phi}\frac{1}{x_{max}-x_{min}}\left(\ln(x_{max}) - \ln(x_{min})\right) & \text{if } \phi x_{max} < x < \psi x_{min} \\ \frac{1}{\psi-\phi}\frac{1}{x_{max}-x_{min}}\left(\ln(\psi x_{max}) - \ln(x)\right) & \text{if } \psi x_{min} < x < \psi x_{max} \end{cases}$$

The risk-neutral segment is absent when $\psi x_{min} = \phi x_{max}$. Thus, his adaptive evaluation $I_X^{U-Weber}(x)$ is convex at the lower end, concave at the upper end, and, possibly, linear in-between.

As Figure 4 shows, for a uniform noise distribution $G(z)$, even if veridical distribution $F(x)$ is uniform (and thus implying risk neutrality), the resulting adaptive evaluation (which is a product convolution truncated to the veridical range) has a complex shape, with risk loving at the lower end, risk neutrality at the middle, and risk aversion at the upper end. Furthermore, small magnitudes tend to be overvalued and large magnitudes tend to be undervalued. For some parameters (Figure 4b) it may even appear to exhibit diminishing marginal utility and overvaluation for most of the reference range,[24]

[24] Compare Figure 4b to the findings on the mental number line by Dehaene, Izard, Spelke, Pica [2008] for numerosity judgments by Munducuru subjects.

consistent with the finding of Burks, Carpenter, Goette and Rustichini [2010] that overvaluation of own skill is much more common than undervaluation.

7 Adding Frequency Imperfections

As Kahneman and Tversky [1979] point out, humans assign non-linear weights to probabilities.[25] Psychologists have accumulated further evidence that human evaluations of proportions follow a very similar type of imperfections (Spence [1990] and Hollands and Dyre [2000]). Here I suggest that, similarly to an imperfect ordinal comparison tool, an imperfect frequency processing tool results in a "twisted" adaptive evaluation, further distorting the relationship between the veridical magnitude distribution and adaptive evaluation, and making it more difficult to observe empirically. The next result is straightforward.

Proposition 9 *Suppose an individual's magnitude evaluation is given by (3) where the perceived magnitude distribution is given by the frequency (proportion) weighting function* $w(F(\tilde{x})$*. Then his adaptive evaluation of magnitude* x *is*

$$I_X^{twisted}(x) = \mathcal{D} * w(F) = \int G(z) dw(F(\tilde{x})) = \begin{cases} \int_X G\left(\frac{x}{\tilde{x}}\right) dw(F(\tilde{x})) & if \quad x = z\ \tilde{x} \\ \int_X G\left(x - \tilde{x}\right) dw(F(\tilde{x})) & if \quad x = z + \tilde{x} \end{cases} \quad (14)$$

That is, the adaptive evaluation is determined by three things: by the context of evaluation $F(\cdot)$, by the ordinal noise $G(\cdot)$ and by the frequency (proportion) weighting $w(\cdot)$. If the ordinal tool is perfect, the following holds.

Corollary 2 *Suppose an individual's magnitude evaluation is given by (14). If he possesses a perfect ordinal comparison tool, then his adaptive evaluation of magnitude* x *is entirely determined by its rank in the twisted (reweighted) magnitude distribution:*

$$I^{twisted}(x) = w(F(x)) \quad (15)$$

As Figures 3 and 4 show, an adaptive evaluation which is subject to both imperfect ordinal tool as well as imperfect frequency tool (e.g. subject to Prelec [1998] frequency imperfections) typically is very different from the veridical distribution. In particular, frequency imperfections tend to boost the adaptive evaluations for relatively small magnitudes and depress evaluations for of relatively large magnitudes. Obviously, as long as at least one of the cognitive tools is imperfect, an adaptive evaluation is neither optimal in a sense of Robson [2001], nor lead to welfare neutrality of economic processes (7).

[25] See Hsu, Krajbich, Zhao and Camerer [2009] for a comprehensive list of probability weighting models.

8 Conclusions

This paper provides a conceptual framework which allows one to combine the insights from economics on one hand and evolutionary and cognitive sciences on the other. There are a few limitations of the adaptive magnitude evaluation model. First, it concentrates on magnitude evaluation, rather than choice. As Rayo and Becker [2007] pointed out, to make a choice among the alternatives, one needs to compare adaptive evaluations across the alternatives, which may involve further perceptual limitations. Second, the model here describes the evaluation of objects that differ only in one dimension, while to evaluate multidimensional objects humans are likely to use additional tools. Third, it remains to be determined whether there are additional cognitive tools involved in processing of uncertainty, as Andreoni and Sprenger [2010] report on a qualitative difference between certain and uncertain utility. Yet, despite its shortcomings, the present paper highlights the importance of recent advances in cognitive and brain research for understanding of human economic decision making.

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