Marriage as a Rat Race: Noisy Premarital Investments with Assortative Matching

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We study the efficiency of premarital investments when parents care about their child's marriage prospects, in a large frictionless marriage market with nontransferable utility. Stochastic returns to investment ensure that equilibrium is unique. We find that, generically, investments exceed the Pareto-efficient level, unless the sexes are symmetric in all respects. Girls will invest more than boys if their quality shocks are less variable than shocks for boys or if they are the abundant sex. The unique equilibrium in our continuum agent model is the limit of the equilibria of finite models, as the number of agents tends to infinity.

I. Introduction

We study the incentives of parents to invest in their children when these investments also improve their marriage prospects. We assume a frictionless marriage market with nontransferable utility. It has usually been thought that ex ante investments suffer from the holdup problem, since

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a parent will not internalize the effects of such investments in her own child on the welfare of the child's future spouse. However, Peters and Siow (2002) argue that in large marriage markets in which the quality of one's match depends on the level of investment, a parent has an incentive to invest more in order to improve the match of her offspring. They argue that the resulting outcome will be Pareto efficient. This is a remarkable result, since they assume a marriage market without transferable utility. With transferable utility, Cole, Mailath, and Postlewaite (2001) show that in large markets, prices can provide incentives for efficient investment decisions.[1]

In this paper, we argue that the optimism of Peters and Siow (2002) must be somewhat tempered. When the return to investment is deterministic, we show that there is a very large set of equilibria. These include efficient outcomes but also a continuum of inefficient ones. In order to overcome this embarrassment of riches, we propose a model in which the returns to investment are stochastic. This is also realistic: talent risk is an important fact of life.[2] Equilibrium in this model is unique, and we are therefore able to make determinate predictions. The model also allows us to address several questions of normative importance and social relevance. Are investments efficient in the absence of prices? What are the implications of biological or social differences between the sexes for investment decision? What are the implications of sex ratio imbalances in countries such as China? Wei and Zhang (2011) argue that marriage market competition for scarce women underlies the high savings rate in China.

Our paper is related to the literature on matching tournaments or contests. This literature typically models a situation in which there is a fixed set of prizes, and agents on the one side of the market compete by making investments, with prizes being allocated to agents according to the rank order of their investments (see e.g., Cole, Mailath, and Postlewaite 1992; Hopkins and Kornienko 2004, 2010). If the "prizes" derive no utility from these investments, for example, when the prize is social status, then an agent's investment exerts a negative positional externality on the other side of the market, so that there is overinvestment. On the other

[1] To appreciate the degree of transferability required, note that Mailath, Postlewaite, and Samuelson (2013) show that one needs "personalized prices," which depend on buyer characteristics as well as seller characteristics, in order to ensure efficiency of investments. Felli and Roberts (2016) show that even in large finite markets, the holdup problem may not disappear if the specificity of investments does not vanish.

[2] Recent studies of the intergenerational transmission of wealth, in the tradition of Becker and Tomes (1979), find an intergenerational wealth correlation of .4 in the United States, which is far from one.

hand, if the prizes derive utility from these investments—for example, if men compete for a set of women with fixed qualities, or students compete for university places—then either overinvestment or underinvestment is possible, depending on how much these investments are valued (Cole et al. 2001; Hopkins 2012).

In our context, investments are two-sided: the investments of men are valued by women and, symmetrically, the investments of women are valued by men. Men do not care directly about how their investments are valued by women; they care only about the consequent improvement in match quality that they get. Women are in a similar situation, since they care only about the improvement in the quality of men that they might get. One might expect, therefore, that this could give rise to underinvestment or overinvestment, depending on parameter values.

Surprisingly, our model yields clear conclusions. Under very special circumstances, when the sexes are completely symmetric, with identical distributions of shocks and a balanced sex ratio, investments will be efficient—not merely in the Pareto sense but also from a utilitarian standpoint. However, if there are any differences between the sexes, whether it be differing returns to investments, different stochastic shocks, or an unequal sex ratio, investments are generically excessive as compared to Pareto-efficient investments. Since the intuition for the overinvestment result is somewhat subtle and quite distinct from that in one-sided tournaments with positional externalities, we defer explaining this until the model is introduced.

The rest of the paper is set out as follows. Section II discusses the problems that arise in a model with deterministic returns and other related literature. Section III sets out the model with noisy investments and shows that a pure strategy equilibrium exists and is unique for general quality functions. We then consider, in turn, additive and multiplicative shocks. Our main finding is that investments are generically excessive, relative to Pareto efficiency. We use our model to examine the observational implications of gender differences and show that if talent shocks are more dispersed for boys than for girls, then girls will invest more than boys. We also examine the effects of sex ratio imbalances on investments and show that the more abundant sex invests more (otherwise, in most of the paper, we focus on the case of a balanced sex ratio). Section IV shows that when there are no gender differences, then investments are efficient, even if there is heterogeneity within each sex. Section V provides a finite agent justification for the continuum model that forms the bulk of the paper. We examine a model with finitely many agents, where there is uncertainty as to whether men will be in excess or women will be in excess. If the number of agents is large enough, there is a unique equilibrium that converges to the equilibrium of the continuum model. Section VI presents conclusions. The Appendix contains proofs that are omitted in the text.

II. Motivation and Related Literature

The fundamental problem is the following: investment in a child benefits the child's future spouse, but the benefit to the spouse is not considered by the child's parents. There is therefore a gap between the *privately optimal* investment in a child, which we denote $\bar{x}$, and the *socially optimal* level, which is naturally greater. In the absence of prices, it is not clear that there are incentives for efficient investment. Peters and Siow (2002) argue that, nonetheless, equilibrium investments are socially efficient.

Let us consider the Peters-Siow model of investment with deterministic returns but simplify by assuming that families are identical rather than differing in wealth. Assume a unit measure of boys, all of whom are ex ante identical, and an equal measure of girls, who are similarly ex ante identical. Assume that the quality of the child, as assessed by a partner in the marriage market, equals the level of parental investment, x. Suppose that a boy is matched with a girl. The utility of the boy's parents is increasing in the investment level of the girl x_G, but they have to bear the cost of investment x_B in their son. Thus, if they choose x_B purely to maximize their utility, they would choose only the privately optimal investment $\bar{x}_B$. But the resulting investment levels $(\bar{x}_B, \bar{x}_G)$ are inefficient. Both families would be better off if they each raised their investments.

Suppose that the family of the boy believes that if they choose investment level x_B, the quality of their son's partner is given by a smooth, strictly increasing function, $\phi(x_B)$. They would choose investments to maximize their overall payoff, given the return function ϕ. Suppose also that the family of a girl believe that the match quality of their girl is also an increasing function of their own investment level x_G. Assume further that this return function equals $\phi^{-1}(x_G)$, the inverse of that for the boys.[3] Consider a profile of investments (x_B^{**}, x_G^{**}) such that x_B^{**} maximizes the payoffs of the boy's family given returns $\phi(x_B)$, x_G^{**} maximizes the payoffs of the girl's family given returns $\phi^{-1}(x_G)$, and $x_B^{**} = \phi^{-1}(x_G^{**})$; that is, these expectations are actually realized. As Peters and Siow argue, the profile (x_B^{**}, x_G^{**}) must be such that the indifference curves on the two sides of the market are mutually tangent, so that the investment profile must be Pareto efficient.

A problem with this approach is that, while the expectations $\phi(x_B)$ are realized in equilibrium, they cannot be realized if the family of a boy chooses $x_B \neq x_B^{**}$. In particular, if a boy deviates and chooses $x_B < x_B^{**}$, the match $\phi(x_B) < x_G^{**}$ is not feasible since every girl in the market has quality x_G^{**}. In other words, while expectations are "rational" at the equi-

[3] If the return function for the girls is the inverse of that for the boys, then whenever a boy raises his investment from x_B to x'_B and finds that his partner's quality rises from x_G to x'_G, it must also be the case that when a girl raises her investment from x_G to x'_G, her partner's quality increases from x_B to x'_B. This property ensures efficiency of investments.

librium, they are not so for any investment level that is not chosen in equilibrium.

We shall be explicit in this paper about the matching process that follows a profile of investments. Specifically, we will require the matching to be feasible, to be stable (in the sense of Gale and Shapley [1962]), and to be measure preserving. Despite these restrictions on matching off the equilibrium path, in terms of equilibria there is an embarrassment of riches and a large set of equilibria. Let (x_B, x_G) be a pair of investments that are weakly greater than the individually optimal investments $(\bar{x}_B, \bar{x}_G)$ and in which the payoff of gender i from being matched with a partner with investment level x_j is weakly greater than the payoff from choosing the individually optimal investment level $\bar{x}_i$ and being unmatched. Any such pair can be supported as an equilibrium by specifying that any agent who deviates to a lower investment level will be left unmatched. In the Peters-Siow equilibrium, an individual making an efficient investment can expect to be matched with someone who invests similarly; there is in effect symmetry. However, here, if the parent of a boy deviates upward and chooses a higher level of investment, his son cannot realize a higher match quality, since all the girls are choosing x_G. We therefore have a “folk theorem”: any pair of investments satisfying the above conditions is an equilibrium. Efficient investments are an equilibrium, but so are inefficient ones.[4]

Turning now to the original Peters-Siow environment in which families differ in wealth, and thereby in their marginal costs of investment, we still find a continuum of inefficient equilibria. The equilibria we have constructed in the homogeneous case are strict equilibria: any individual who invests differently does strictly worse. If we perturb wealth levels slightly and wealth affects payoffs continuously, then these equilibria will continue to be strict. The only thing that is required is that the distribution of wealth is not too dispersed, so that there is a common level of investment $\hat{x}$ that is not so low that it is below the richest family’s privately optimal investment and not so high that the poorest family would prefer to deviate downward and be unmatched. None of these equilibria are efficient. In fact, for all of them, a measure 0 of agents make an efficient investment. If $\hat{x}$ is relatively low, then all agents underinvest. If $\hat{x}$ is higher, some agents underinvest and some overinvest. One can also construct inefficient equilibria, with a heterogeneity of investment levels, even when wealth is more widely dispersed.[5]

[4] Equilibrium multiplicity also holds if the deviator is left unmatched with probability one-half rather than for sure. This matching rule can be justified in a large finite model (see Sec. V).

[5] For example, we may divide families into two groups, rich and poor, each of which has a common level of investment. The matching rule matches those families with sons who choose investment $\hat{x}_L$ to those families with daughters with the same investment and matches those who choose $\hat{x}_H$ to daughters with the same investment.

Our paper shows that these problems can be resolved if we augment the model by adding an idiosyncratic element of match quality. This ensures that there is always a nondegenerate distribution of qualities on both sides of the market, thereby providing incentives to invest. Furthermore, equilibrium is unique under some regularity conditions. Before proceeding to the model, we review some of the related literature not already discussed in the introduction.

Peters (2007) investigates two-sided investments with finitely many agents. He assumes that individuals on the long side of the market may drop out of the market with some small probability and solves for an equilibrium in mixed strategies.[6] Peters (2009) assumes that there is ex ante heterogeneity rather than the noisy returns assumed in this paper. In both papers, equilibrium investments are bounded away from the efficient level even as the number of participants goes to infinity.

Hoppe, Moldovanu, and Sela (2009) analyze a signaling model of matching in which an agent cares about his or her match partner's underlying characteristic, which is private information. Again there is ex ante heterogeneity rather than stochastic returns. Since investments are not directly valued, they are inherently wasteful, although they may improve allocative efficiency in the matching process. They also obtain interesting comparative statics results, on gender differences and on the numbers of participants, that we relate to our own results. Hopkins (2012) finds that with one-sided investments, the level of investment can be inefficiently low.

Our approach differs from most of the theoretical literature on investments, which usually assumes ex ante heterogeneity or incomplete information (Hoppe et al. 2009; Peters 2009; Hopkins 2012). Agents are assumed to differ ex ante in terms of quality or wealth, giving rise to heterogeneity in investments. Instead, we build on the classic work of Lazear and Rosen (1981), who analyze a tournament in which a finite number of identical workers compete for exogenously given prizes. By assuming that a worker's output is noisy, they ensure that the optimization problem faced by the worker is smooth. Models with noisy returns face the difficulty that the optimization problem faced by the agent is not necessarily concave. By assuming that the noise is large enough, Lazear and Rosen ensure that the effort level that satisfies the first-order condition is also globally optimal. With two-sided investments, our problem is somewhat more delicate, since it is the *relative dispersion* that matters. Increasing dispersion on one side, say men, increases the payoff to large deviations on the women's side. One of our contributions is to show how this

[6] This works in a similar way to the uncertainty over participant numbers assumed in Sec. V of the current paper.

analysis may be extended to two-sided matching and investments and to a situation in which the number of agents is large.

Gall, Legros, and Newman (2009) also employ a model with noisy returns to investments and examine investments, matching, and affirmative action in a nontransferable utility setting. They consider a situation in which efficiency requires negative assortative matching but in which stable matchings are positively assortative, providing a possible rationale for affirmative action. They allow for investments with stochastic returns and focus on the trade-off between the positive role of affirmative action on match efficiency versus its possible negative effect on investment incentives.

In empirical work on matching, typically the value of any match is assumed to have an idiosyncratic random element (Dagsvik 2000; Choo and Siow 2006). Our key finding here is that the structure of shocks not only affects the matching process but is also critical for investment incentives.

The transferable utility model is an alternative paradigm, pioneered by Becker (1973), that can be used to explain several empirical phenomena. Chiappori, Iyigun, and Weiss (2009) use this model to explain the increasing education of women, while Iyigun and Walsh (2007) study the distributional consequences of institutional and gender differences for investments. While transferable utility models are very useful, there are many reasons for the limited transferability of utility in the marriage context, including the inability to commit to future transfers at the time of marriage.[7] Finally, there is also work that considers investment incentives in the presence of search frictions: Acemoglu and Shimer (1999) study one-sided investments under transferable utility, while Burdett and Coles (2001) analyze a nontransferable utility model with two-sided investments.

III. A Matching Tournament with Noisy Investments

We now set out a model in which the returns to investments are stochastic. To simplify the analysis, we assume that there is no ex ante heterogeneity but there are stochastic returns so that there is heterogeneity ex post. Thus, all families are ex ante identical, save for the fact that some have boys and others have girls. We assume a balanced sex ratio and a continuum population, so that there are equal measures of boys and girls. Under these assumptions and under certain technical conditions, a pure strategy equilibrium exists and is unique.

Assume that a parent of a boy chooses an investment x in a bounded interval $[0, \bar{x}_B)$ and derives a direct private benefit $b_B(x)$ and incurs a cost

[7] Gall et al. (2009) provide reasons why in many matching situations, including business partnerships, transfers between potential partners may not be fully flexible.

$\tilde{c}_B(x)$. Similarly, a parent of a girl chooses x in a bounded interval $[0, \tilde{x}_G)$ and derives a direct private benefit $b_G(x)$ and a cost $\tilde{c}_G(x)$, respectively. Define the net cost of investment in a child of gender i, $i \in \{G, B\}$, as $c_i(x) \coloneqq \tilde{c}_i(x) - b_i(x)$. The quality of a child is an increasing function of the level of parental investment, x, and the realization of a random shock. These functions are written as $q^B(x, \varepsilon)$ for boys and $q^G(x, \eta)$ for girls. The first argument in each quality function denotes parental investment, and the second argument denotes the random shock.

A parent of any boy believes that her son's shock, denoted ε, is distributed with a density function $f(\varepsilon)$ and a cumulative distribution function (cdf) $F(\varepsilon)$. Similarly, the parent of a girl believes that the daughter's shock, η, is distributed with a density function $g(\eta)$ and cdf $G(\eta)$. The aggregate realized distribution of shocks in the population of boys (respectively, girls) is deterministic and equals F (respectively, G). Since our model does not require a continuum of independent and identically distributed (i.i.d.) random variables, there exists a simple probabilistic model consistent with these assumptions.[8]

Our technical assumptions are as follows.

Assumption 1.

1. Assumptions on shocks: let $f(\varepsilon)$ and $g(\eta)$ be twice continuously differentiable on their bounded supports $[\underline{\varepsilon}, \bar{\varepsilon}]$ and $[\underline{\eta}, \bar{\eta}]$, respectively. Further, assume that $f(\underline{\varepsilon}) = 0$, $g(\underline{\eta}) = 0$, but $f(\varepsilon)$ and $g(\eta)$ are otherwise strictly positive on their supports. Assume that the right-hand derivatives at the lower bound of their supports, denoted as $f'(\underline{\varepsilon})$ and $g'(\underline{\eta})$, are strictly positive.
2. Assumptions on net costs: for $i \in \{G, B\}$, $c_i(x)$ is twice continuously differentiable on the open interval that contains the set of feasible investments, $[0, \tilde{x}_i)$, and satisfies the following: (*a*) convexity: $c_i''(\cdot)$ is bounded below on $[0, \tilde{x}_i)$ by $\gamma > 0$; (*b*) $c_i'(0)$ is strictly negative, and $\lim_{x \to \tilde{x}_i} c_i'(x) = \infty$.
3. Assumptions on quality: let I_ε and I_η be open intervals that contain $[\underline{\varepsilon}, \bar{\varepsilon}]$ and $[\underline{\eta}, \bar{\eta}]$, respectively, and $q^B : \mathbb{R}^+ \times I_\varepsilon \to \mathbb{R}$ and $q^G : \mathbb{R}^+ \times I_\eta \to \mathbb{R}$ be increasing and twice differentiable, with $q_x^B(x_B, \varepsilon) > 0$; $q_x^G(x_G, \eta) > 0$; $q_\varepsilon^B(x_B, \varepsilon) > 0$ if $x_B > 0$; $q_\eta^G(x_G, \eta) > 0$ if $x_G > 0$; $q_{xx}^B(x_B, \varepsilon) \le 0$; $q_{\varepsilon\varepsilon}^B(x_B, \varepsilon) = 0$; $q_{xx}^G(x_G, \eta) \le 0$; $q_{\eta\eta}^G(x_G, \eta) = 0$; $q_{x\varepsilon}^B(x_B, \varepsilon) \ge 0$; and $q_{x\eta}^G(x_G, \eta) \ge 0$.[9]

[8] Normalize the Lebesgue measure of boys to 1, and let their index be uniformly distributed on [0, 1]. Fix an arbitrary individual, say 0, and draw his shock value according to F. Let $z(0)$ denote the realization of this draw. For a boy of arbitrary index $t \in (0, 1]$, his shock value $z(t)$ equals $F^{-1}[F(z(0)) + t]$ if $F(z(0)) + t \le 1$ and equals $F^{-1}[F(z(0)) + t - 1]$ otherwise. Thus for every t, $z(t)$ is distributed according to F, and the aggregate distribution also equals F.

[9] That is, quality is assumed to be concave in investment x but linear in shocks ε, η. These assumptions allow for the additive and multiplicative specifications as special cases.

4. The value of not being matched is $\bar{u}$, which satisfies $\bar{u} < q^B(0, \underline{\varepsilon})$ and $\bar{u} < q^G(0, \underline{\eta})$. That is, a girl who invests x and who is not matched has total payoff $\bar{u} - c_G(x)$.

The final point in assumption 1 implies that the value from being unmatched, $\bar{u}$, is strictly less than the payoff from the lowest possible quality match. Also, note that, matched or unmatched, the individual still pays the investment cost.

Parents are altruistic and internalize the effects of their decisions on the utility of their own child, but not on the utility of their child's partner. Thus if a girl with parental investment x_G and shock η is matched with a boy whose parent has invested x_B and who has shock realization ε, her payoff and that of her parents equal[10]

$$U_G(x_G, x_B) = q^B(x_B, \varepsilon) + b_G(x_G) - \tilde{c}_G(x_G) = q^B(x_B, \varepsilon) - c_G(x_G). \quad (1)$$

Similarly for a boy of type (x_B, ε) who is matched with a girl of type (x_G, η), his utility would be

$$U_B(x_B, x_G) = q^G(x_G, \eta) + b_B(x_B) - \tilde{c}_B(x_B) = q^G(x_G, \eta) - c_B(x_B). \quad (2)$$

Let $\bar{x}_B$ denote the *individually optimal* investment for boys; it is the investment that minimizes $c_B(x) = \tilde{c}_B(x) - b_B(x)$. Since $c'_B(0) < 0$, the individually optimal investment satisfies $c'_B(\bar{x}_B) = 0$ and $\bar{x}_B > 0$. The individually optimal investment for girls, $\bar{x}_G$, is defined similarly, and $\bar{x}_G > 0$. Individually optimal investments are not Pareto efficient. Consider a social planner who chooses (x_B, x_G) to maximize

$$\begin{aligned} W(x_B, x_G) = \lambda & \left[\int q^B(x_B, \varepsilon) f(\varepsilon) d\varepsilon - c_G(x_G) \right] \\ & + (1 - \lambda) \left[\int q^G(x_G, \eta) g(\eta) d\eta - c_B(x_B) \right], \end{aligned} \quad (3)$$

for some $\lambda \in (0, 1)$, where λ is the relative weight placed on the welfare of girls. Differentiating with respect to x_B and x_G, setting to zero, and rearranging, we obtain the first-order conditions for Pareto efficiency,

$$\frac{c'_B(x_B)}{\int q^B_x(x_B, \varepsilon) f(\varepsilon) d\varepsilon} = \frac{\lambda}{1 - \lambda}, \quad (4)$$

[10] Our analysis also applies when the partner's valuation of quality is an increasing concave function of q^i.

$$\frac{c'_G(x_G)}{\int q_x^G(x_G,\eta)g(\eta)d\eta} = \frac{1-\lambda}{\lambda}. \tag{5}$$

Rearranging the first-order condition for welfare maximization, we obtain

$$c'_B(x_B) \times c'_G(x_G) = \int q_x^B(x_B,\varepsilon)f(\varepsilon)d\varepsilon \times \int q_x^G(x_G,\eta)g(\eta)d\eta. \tag{6}$$

In other words, any profile of Pareto-efficient investments satisfies this condition, irrespective of the value of λ. Pareto-efficient investments always exceed the privately optimal level because under the privately optimal investments, we have $c'_B(\bar{x}_B) = c'_G(\bar{x}_G) = 0$, whereas the right-hand side of equation (6) is strictly positive, since q_x is strictly positive. Of particular interest is the case in which λ, the weight placed on girls' welfare, is equal to their proportion in the population, one-half. Let x_B^{**}, x_G^{**} denote the efficient investments in this case. We shall call these the *utilitarian efficient* investments. These are the investments that parents would like the social planner to choose in the "original position" before the gender of their child is realized.

The Pareto efficiency condition (6) does not determine a unique investment level, but a continuous curve in (x_B, x_G) space. If a profile of investments (x_B, x_G) is such that the product of the marginal costs is strictly greater than the right-hand side of equation (6) and the marginal costs are positive, then such a point lies above the Pareto efficiency curve.[11] We say that we then have *overinvestment relative to Pareto efficiency*, since it is possible to achieve Pareto efficiency by reducing either investment level.[12] Similarly, if the product of marginal costs is strictly less than the right-hand side of the equation, we have *underinvestment relative to Pareto efficiency*.

On the other hand, utilitarian efficiency determines a unique point in (x_B, x_G) space. Thus we may consider the investments of one side alone, say boys, and speak of underinvestment relative to the utilitarian level, without reference to the investments by girls. Thus, by the utilitarian criterion, one may have underinvestment by boys and overinvestment by girls. By the Pareto criterion, one can have only overinvestment or underinvestment, where this statement applies to the profile of investments, (x_B, x_G).

We shall focus on pure strategy Nash equilibria in which every parent on a given side of the marriage market chooses the same level of invest-

[11] No individual will choose investment below the individually optimal level since higher investments can never reduce match quality. Thus we may restrict attention to investment levels such that marginal costs are nonnegative.

[12] This follows from the concavity of $q^i(\cdot)$ in x and the strict convexity of $c_B(\cdot)$ and $c_G(\cdot)$. If we reduce x_B, then the left-hand side of (6) decreases as a result of the strict convexity of the cost function, while the first term on the right-hand side increases, since $q_{xx}^B(x_B,\varepsilon) \le 0$ by assumption 1. Since $c'_B(\bar{x}_B) = 0$, there exists a reduction in x_B such that (6) holds.

ment. Such an equilibrium will be called *quasi-symmetric* and consists of a pair (x_B^*, x_G^*). We require the matching to be stable and measure preserving. Given our specification of preferences, whereby all boys uniformly prefer girls of higher quality, and vice versa, a stable measure preserving matching is essentially unique and must be assortative. Since q is strictly increasing in the idiosyncratic shock (as long as investments are nonzero) and since all agents on the same side of the market choose the same investment level, in equilibrium, there must be matching according to the idiosyncratic shocks alone. Recall that the realized distribution of shocks in the population is deterministic. For a boy who has shock realization ε, let $\phi(\varepsilon)$ denote the value of η of his match. This satisfies

$$F(\varepsilon) = G(\phi(\varepsilon)), \tag{7}$$

or $\phi(\varepsilon) = G^{-1}(F(\varepsilon))$. That is, if a boy is of rank z in the boys' distribution, he is matched with a girl of the same rank z in the girls' distribution. The nondegenerate distribution of qualities on both sides of the marriage market provides incentives of investment above the privately optimal level. If the parent of a boy invests a little more than x_B^*, he increases the boy's rank for any realization of ε. By doing so, he obtains a girl of higher rank. However, he is concerned not with the girl's rank but with her quality.

One delicate issue concerns large deviations from the equilibrium, where the quality realization is outside the support of the equilibrium distribution of qualities. For example, if a boy deviates upward and his quality exceeds $q^B(x_B, \bar{\varepsilon})$, stability implies that he will be matched with the best-quality girl, of quality $q^G(x_G, \bar{\eta})$. If he deviates downward and his quality is below $q^B(x_B, \underline{\varepsilon})$, then stability implies that he could be left unmatched (with payoff $\bar{u}$) or matched with the lowest-quality girl. We shall assume that both these outcomes have equal probability. Since we assume that being single has a low payoff, this deters large downward deviations. These assumptions are consistent with the requirement that the matching be stable and measure preserving. Moreover, the matching assumption can be justified as the limit of a model with a finite number of agents as the number of agents tends to infinity, as we show in Section V, where we consider a model in which the exact numbers of men and women are random: with probability one-half there are slightly more men than women, and with probability one-half the reverse is the case. Then a boy with the lowest quality is unmatched with probability one-half.

In the Appendix we show that the first-order condition for the equilibrium investment in boys can be written as

$$\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} q_\eta^G(x_G, \phi(\varepsilon)) \frac{f(\varepsilon)}{g(\phi(\varepsilon))} \frac{q_x^B(x_B^*, \varepsilon)}{q_\varepsilon^B(x_B^*, \varepsilon)} f(\varepsilon)\, d\varepsilon = c_B'(\hat{x}_B), \tag{8}$$

where $\hat{x}_B$ denotes the "best response" by boys to x_G. The intuition for the first-order condition is that it balances the marginal cost $c_B'(\cdot)$ of extra investment on the right-hand side with its marginal benefit on the left-hand side. The latter principally is determined by the possibility of an improved match from increased investment. Specifically, an increase in ε, a boy's shock, would improve his match, given the matching relation (7) at rate $\phi' = f/g$. Similarly, the first-order condition for investment in girls is given by

$$\int_{\underline{\eta}}^{\bar{\eta}} q_{\varepsilon}^{B}(x_B, \phi^{-1}(\eta)) \frac{g(\eta)}{f(\phi^{-1}(\eta))} \frac{q_x^G(x_G^*, \eta)}{q_{\eta}^G(x_G^*, \eta)} g(\eta)\, d\eta = c_G'(\hat{x}_G), \tag{9}$$

where $\hat{x}_G$ is the best response by girls to x_B. If a profile (x_B^*, x_G^*) is a quasi-symmetric equilibrium, then it must satisfy $x_B^* = \hat{x}_B(x_G^*)$ and $x_G^* = \hat{x}_G(x_B^*)$.

Note that the match value of remaining single, $\bar{u}$, does not affect the first-order condition for equilibrium investments and thus does not affect the equilibrium level. This is so since in equilibrium, an individual is always matched with probability one when the sex ratio is balanced. Furthermore, we assume that the density function of shocks is zero at its lower bound, ensuring that $\bar{u}$ does not affect the derivative of the payoff function at equilibrium. However, the value of $\bar{u}$ does affect the payoff from large downward deviations, and we assume a "misery effect," that is, that $\bar{u}$ is sufficiently small relative to the payoff from being matched.[13] This ensures that large downward deviations are not profitable.

We also have to ensure that large upward deviations are not profitable; this is not immediate, since the optimization problem faced by agents is not necessarily quasi-concave, just as in Lazear and Rosen (1981).[14] We therefore invoke the following assumption.

Assumption 2. One of the following two conditions is satisfied:

a. F and G are distributions of the same type, that is, $G(x) = F(ax + b)$;
b. $f(\varepsilon)$ and $g(\eta)$ are weakly increasing.

Assumption 2 ensures that the benefit function is concave for upward deviations and, together with assumption 1, ensures existence of a quasi-symmetric equilibrium in pure strategies. For uniqueness of quasi-symmetric equilibrium, we invoke the following additional assumption.

[13] The total payoff to a boy from being single equals $\bar{u} - c_B(x_B)$. This incorporates the private benefit from investment, since $c_B(x_B) = \tilde{c}_B(x_B) - b_B(x_B)$ measures the cost net of the private benefit.

[14] In a one-sided tournament it suffices to assume that shocks affecting that side are sufficiently dispersed and that the cost function is convex. In a two-sided tournament, it is the relative dispersion on the two sides that matters, and one cannot make shocks relatively dispersed for both sides. See the discussion of the lognormal example in Sec. III.A.3.

ASSUMPTION 3. Quality is *generalized additive/multiplicative*:

$$q^B(x_B, \varepsilon) = \theta_B(x_B + \varepsilon) + (1 - \theta_B)\gamma(x_B)\varepsilon,$$

$$q^G(x_G, \eta) = \theta_G(x_G + \eta) + (1 - \theta_G)\gamma(x_G)\eta,$$

where $\theta_i \in [0, 1]$, for $i \in \{G, B\}$, and $\gamma(\cdot)$ is strictly increasing, twice differentiable, and strictly concave, with $\gamma(0) = 0$.

THEOREM 1. Under assumptions 1 and 2 and if the value to not being matched ($\bar{u}$) is sufficiently low, there exists a quasi-symmetric Nash equilibrium of the matching tournament. Under assumption 3, the quasi-symmetric equilibrium is unique.

Now that existence has been established, we can turn to some important questions about the qualitative nature of equilibrium behavior. Are investments Pareto efficient? What are the factors that lead one sex to invest more than the other? In order to shed more light on these questions, we consider, in turn, different specifications of the quality functions $q^i(\cdot)$.

A. *Additive Shocks*

We first analyze the case in which $q^B(x, \varepsilon) = x + \varepsilon$ and $q^G(x, \varepsilon) = x + \eta$. One interpretation is that investments or bequests are in the form of financial assets or real estate, while the shocks are to (permanent) labor income of the child. The interpretation is that total household income is like a public good (as in Peters and Siow [2002]), which both partners share.

Consider a quasi-symmetric equilibrium in which all boys invest x_B^* and all girls invest x_G^*. A boy with shock realization ε and of rank z in the distribution $F(\cdot)$ will be matched with a girl of shock $\phi(\varepsilon)$ with the same rank z in $G(\cdot)$. Suppose that the parent of a boy invests a little more, $x_B^* + \Delta$, as in figure 1. If his realized shock is ε, the improvement in the ranking of boys is approximately equal to $f(\varepsilon)\Delta$. The improvement in the quality of the matched girl, $\tilde{\Delta}$, must be such that $g(\eta)\tilde{\Delta} \approx f(\varepsilon)\Delta$; that is, the improvement in the rank of his match must equal the improvement in his own rank. Thus the marginal return to investment in terms of match quality equals $f(\varepsilon)/g(\phi(\varepsilon))$ at any value of ε.

Integrating over all possible values of ε gives the first-order condition for optimal investments in boys,

$$\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \frac{f(\varepsilon)}{g(\phi(\varepsilon))} f(\varepsilon)\, d\varepsilon = c_B'(x_B^*). \tag{10}$$

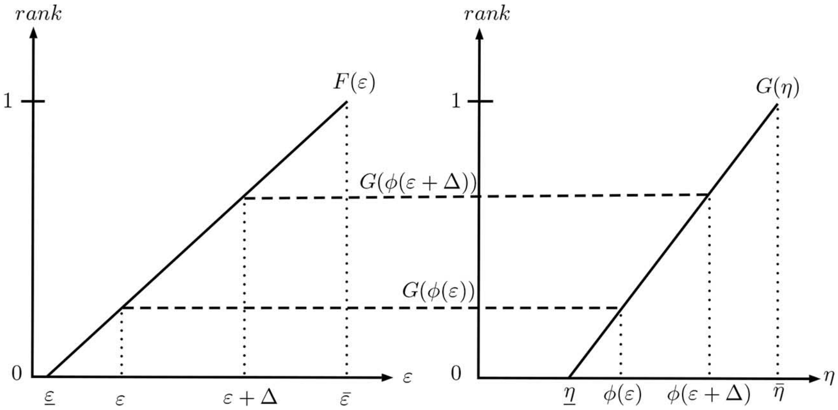

FIG. 1.—If a boy with shock realization ε increases investment by an amount Δ, he would overtake other boys with shock realizations between ε and $\varepsilon + \Delta$. The boy's match would improve from the girl with shock value $\phi(\varepsilon)$ to one at $\phi(\varepsilon + \Delta)$.

Similarly, the first-order condition for investment in girls is

$$\int_{\underline{\eta}}^{\bar{\eta}} \frac{g(\eta)}{f(\phi^{-1}(\eta))} g(\eta)\, d\eta = c'_G(x^*_G). \tag{11}$$

The left-hand side of the above equations (the marginal benefit) is constant, while the right-hand side is strictly increasing in x_i because of the convexity of the cost function. Thus, there is a unique solution to the first-order conditions.

1. Efficiency

We now use the first-order conditions (10) and (11) to examine the efficiency of investments. Under additive shocks, the marginal benefit to a girl from a boy's investment is one, regardless of the realization of the shock. Thus the Pareto efficiency conditions (4) and (5) reduce to $c'_B(x_B) = \lambda/(1-\lambda)$ and $c'_G(x_G) = (1-\lambda)/\lambda$. This implies that in any Pareto-efficient allocation, $c'_B(x_B) \times c'_G(x_G) = 1$. The condition for utilitarian efficiency (where equal weight is placed on the welfare of boys and girls) is $c'_B(x_B) = c'_G(x_G) = 1$.

Suppose $F = G$, that is, the distribution of shocks is the same. Thus, $f(\varepsilon)/g(\phi(\varepsilon)) = 1$ for all values of ε, so that $c'_B(x^*_B) = c'_G(x^*_G) = 1$. Investments are utilitarian efficient even if the investment cost functions are different for the two sexes. As we shall see later, this is an example of a more general result: if there are no gender differences whatsoever, this ensures utilitarian efficiency. In general, if there are any differences between the sexes, $f(\varepsilon)/g(\phi(\varepsilon))$ will differ from one, and so one cannot expect utilitarian efficiency. The following theorem sharpens this conclusion.

Theorem 2. When noise is additive, in a quasi-symmetric equilibrium, investments are generically excessive relative to Pareto efficiency.

Proof. It is useful to make the following change in variables in the first-order condition for the girls, (11). Since $\eta = \phi(\varepsilon)$,

$$d\eta = \phi'(\varepsilon)d\varepsilon = \frac{f(\varepsilon)}{g(\phi(\varepsilon))}d\varepsilon.$$

Thus the first-order condition for girls is rewritten as

$$\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} g(\phi(\varepsilon))d\varepsilon = c'_G(x^*_G).$$

Consider the product of the two first-order conditions:

$$c'_B(x^*_B) \times c'_G(x^*_G) = \left[\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \frac{f(\varepsilon)}{g(\phi(\varepsilon))} f(\varepsilon)d\varepsilon\right]\left[\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} g(\phi(\varepsilon))d\varepsilon\right].$$

By the Cauchy-Schwarz inequality,

$$\left[\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \frac{f(\varepsilon)}{g(\phi(\varepsilon))} f(\varepsilon)d\varepsilon\right]\left[\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} g(\phi(\varepsilon))d\varepsilon\right]$$
$$\geq \left(\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \left\{\frac{f(\varepsilon)}{[g(\phi(\varepsilon))]^{1/2}}\right\}[g(\phi(\varepsilon))]^{1/2}d\varepsilon\right)^2 = 1,$$

with the inequality being strict if the two terms are linearly independent. Thus $c'_B(x^*_B) \times c'_G(x^*_G) > 1$ if $f(\varepsilon)/\sqrt{g(\phi(\varepsilon))}$ and $\sqrt{g(\phi(\varepsilon))}$ are linearly independent functions of ε. Since Pareto efficiency requires $c'_B(x^*_B) \times c'_G(x^*_G) = 1$, we have overinvestment generically if the distributions f and g differ. QED

Example 1. Let us assume that $F(\varepsilon) = \varepsilon$ on $[0, 1]$, that is, ε is uniformly distributed.[15] Assume that $G(\eta) = \eta^n$ on $[0, 1]$. Then $F(\varepsilon) = G(\phi(\varepsilon))$ implies $\phi(\varepsilon) = \varepsilon^{1/n}$ and $g(\phi(\varepsilon)) = n\varepsilon^{(n-1)/n}$. The equilibrium conditions are

$$c'_B(x^*_B) = \int_0^1 \frac{f(\varepsilon)}{g(\phi(\varepsilon))} f(\varepsilon)d\varepsilon = \frac{1}{n}\int_0^1 \varepsilon^{(1-n)/n}d\varepsilon = 1,$$

[15] The uniform distribution violates our assumption 1, since $f(\underline{\varepsilon}) > 0$. This implies that the left-hand derivative of a boy's payoffs with respect to investments is strictly greater than the right-hand derivative (see fn. 21 in the Appendix). We focus on the equilibrium in which the right-hand derivative equals zero, i.e., the one with smallest investments. Any other equilibrium will have strictly larger investments.

$$c'_G(x^*_G) = \int_0^1 \frac{g(\eta)}{f(\phi^{-1}(\eta))} g(\eta) d\eta = n^2 \int_0^1 \eta^{2n-2} d\eta = \frac{n^2}{2n-1}.$$

The product of the marginal costs equals $n^2/(2n-1) > 1$ for $n > ½$ and $n \neq 1$. Efficiency requires that the product equals one, which it does only for $n = 1$, that is, when $f = g$.

The example provides additional intuition for the inefficiency result. Let $n = 2$, so that the density function for women $g(\eta) = 2\eta$ on $[0, 1]$. The incentive for investment for a man at any value of ε depends on the ratio of the densities, $f(\varepsilon)/g(\phi(\varepsilon))$. This ratio exceeds one for low values of ε but is less than one for high values of ε. Conversely, for women, the incentive to invest depends on the inverse of this ratio, $g(\eta)/f(\phi^{-1}(\eta))$, which is low at low values of η but high at high values of η. In other words, the ratio of the densities plays opposite roles for the two sexes. However, the weights with which these ratios are aggregated differ between the sexes; high values of η are given relatively large weight in the case of women, since $g(\eta)$ is large in this case, while they are given relatively less weight in the case of men.

2. Gender Differences

We use our model to examine a contentious issue: what are the implications of gender differences? Let us assume that the shocks to quality constitute talent shocks and that quality is additive in talent and investment (our results in this section also apply when quality is multiplicative). One issue that excites great controversy is whether the distributions differ for men and women. For example, Baron-Cohen (2003) and Pinker (2008) argue that there are intrinsic gender differences that are rooted in biology, while Fine (2008) has attacked this view. In any case, in a study based on test scores of 15-year-olds from 41 OECD countries, Machin and Pekkarinen (2008) find that boys show greater variance than girls in both reading and mathematics test scores in most countries. We now explore the implications of differences in variability between the sexes.

Suppose that the shocks are more variable for men than for women. One way to formalize the idea of a distribution being more variable than another is the dispersive order. A distribution F is larger in the dispersive order than a distribution G, or $F \geq_d G$ if

$$g(G^{-1}(z)) \geq f(F^{-1}(z)) \quad \text{for all } z \in (0,1), \tag{12}$$

with the inequality being strict on a set of z values with positive measure (see Shaked and Shanthikumar 2007, 148–49). For example, if F and G are both uniform distributions, where the support of F is a longer interval than that of G, then $F \geq_d G$. A second example is two normal distributions; the one with the higher variance is larger in the dispersive order. These measures of dispersive order do not rely on an equality of means (see Hopkins and Kornienko [2010] for further examples and discussion).

Suppose that $F \geq_d G$, so that $f(\varepsilon)/g(\phi(\varepsilon)) \leq 1$ for all values of ε and is strictly less on a set of values of ε of positive measure. Thus the integral on the left-hand side of equation (10) is strictly less than one, and the integral on the left-hand side of equation (11) is strictly greater than one. As utilitarian efficiency requires $c_B'(x_B) = 1 = c_G'(x_G)$, boys underinvest and girls overinvest, relative to the utilitarian level. We therefore have the following proposition.

Proposition 1. With additive shocks, if the distribution of shocks for boys is more dispersed than that for girls, that is, $F \geq_d G$, then there is underinvestment in boys and there is overinvestment in girls relative to the utilitarian efficient level.

The intuition for this result is as follows. If the distribution of shocks for boys is relatively dispersed, then at any realization of ε, an increment in his investment results in only a small improvement in his rank, and thus of his partner. Since the quality of the girls is relatively compressed, this improvement in the rank of his partner translates to only a small increase in quality. In contrast, for a girl, an increment in investment results in a large improvement in her rank, and this improvement in the rank of her partner also translates to a large increase in quality, given the higher dispersion in boy qualities. Therefore, in equilibrium, investment in girls is greater than in boys.

Empirically, the average performance of girls in school is often better than that of boys, especially in developed countries, where there is less discrimination. Our model provides a possible partial explanation for this: the incentives to invest for girls are greater, from marriage market matching considerations. While differences in the dispersion of shocks have strong implications, differences in the mean play no role in investment incentives. To see this, suppose that f is a translation of g, that is, $f(\varepsilon) = g(\varepsilon + k)$ for some k. This implies that $\phi(\varepsilon) = \varepsilon + k$, so that $f(\varepsilon)/g(\phi(\varepsilon)) = 1$ for every ε. Investments will be utilitarian efficient, and this difference in average quality has no implications for investment incentives.

3. Normal or Lognormal Shocks

Suppose that the shocks are normally distributed, that is, $\varepsilon \sim N(\mu_\varepsilon, \sigma_\varepsilon)$ and $\eta \sim N(\mu_\eta, \sigma_\eta)$. Thus $F(\varepsilon) = \Phi((\varepsilon - \mu_\varepsilon)/\sigma_\varepsilon)$ and $G(\eta) = \Phi((\eta - \mu_\eta)/\sigma_\eta)$, where Φ denotes the standard normal cdf. Thus the matching $\phi(\varepsilon)$ is linear, and $f(\varepsilon)/g(\phi(\varepsilon)) = \sigma_\eta/\sigma_\varepsilon$ at all values of ε. Furthermore, linearity of the matching implies that $\phi'(\varepsilon + \Delta)$ is constant and equal to $\sigma_\eta/\sigma_\varepsilon$, implying that agents' optimization is strictly concave as long as the cost function is convex. Since shocks are unbounded, an agent is always matched even when he deviates downward, and the misery effect plays no role in deterring downward deviations. The first-order conditions for investment are

$$c_B'(x_B^*) = \frac{\sigma_\eta}{\sigma_\varepsilon}, \qquad c_G'(x_G^*) = \frac{\sigma_\varepsilon}{\sigma_\eta}.$$

Investments are always Pareto efficient but will not be utilitarian efficient if the variances differ. If one measures the degree of over- or underinvestment relative to the utilitarian level by the associated marginal costs and if $\sigma_\varepsilon > \sigma_\eta$, then the overinvestment by girls, relative to the utilitarian level, is proportional to the ratio of the standard deviations.

Our analysis can be extended to the case in which an increasing function of the shocks is normally distributed. For example, consider lognormal shocks, so that $\ln(\varepsilon) \sim N(0, \sigma_\varepsilon)$ and $\ln(\eta) \sim N(0, \sigma_\eta)$. The first-order conditions for investment (see the Appendix for derivation) are given by

$$\frac{\sigma_\eta}{\sigma_\varepsilon}\exp\left[\frac{1}{2}(\sigma_\eta - \sigma_\varepsilon)^2\right] = c_B'(x_B^*),$$

$$\frac{\sigma_\varepsilon}{\sigma_\eta}\exp\left[\frac{1}{2}(\sigma_\eta - \sigma_\varepsilon)^2\right] = c_G'(x_G^*).$$

As in the normal case, the ratio of the marginal costs is related to the ratio of the variances. However, here we have

$$c_B'(x_B^*) \times c_G'(x_G^*) = \exp(\sigma_\eta - \sigma_\varepsilon)^2,$$

so that the outcome is neither utilitarian nor even Pareto efficient unless $\sigma_\varepsilon = \sigma_\eta$. The extent of inefficiency is related to the difference in variances.

Finally, note that our general existence theorem does not apply to these examples since it assumes bounded shocks. In the normal case, linearity of the matching suffices to ensure that the maximization problem is strictly concave. The Appendix shows that large upward deviations are not profitable in the lognormal case under plausible assumptions on the ratio of variances and the convexity of costs. Thus our analysis can be extended more generally, beyond the class of distributions satisfying assumption 2, if one assumes explicit forms for the distribution of shocks.

4. Sex Ratio Imbalances

Sex ratio imbalances are an important phenomenon in countries such as China and parts of India. These imbalances are extremely large in China, where it is estimated that one in five boys born in the 2000 census will be unable to find a marriage partner (see Bhaskar 2011). Wei and Zhang

(2011) argue that the high savings rate in China is partly attributable to the sex ratio imbalance. They argue that parents of boys feel compelled to invest more in order to improve their chances of finding a partner, thus raising the overall savings rate. However, one might conjecture that this might be counterbalanced by the reduced pressure felt by the parents of girls. We therefore turn to our model to provide an answer to this question.

Assume that each sex is ex ante identical, and let the relative measure of girls equal $r < 1$. At the matching stage, since $r \leq 1$, all girls should be matched, and the highest-quality boys should be matched. Since every girl is matched, the investment in her generates benefits for herself and for her partner (for sure). Thus the first-best investment level in a girl, x_G^{**}, satisfies $c_G'(x_G^{**}) = 1$. Now consider investment in a boy. If we assume that the idiosyncratic component of match values is sufficiently small, then welfare optimality requires that only a fraction r of boys invest and that their investments also satisfy $c_B'(\cdot) = 1$. However, if we restrict attention to symmetric investment strategies, then investment will take place in all boys; and since investment occurs before ε is realized, each boy has a probability r of being matched. Thus the utilitarian efficient level of investment in a boy, x_B^{**}, must satisfy $c_B'(x_B^{**}) = r$; that is, the marginal cost must equal the expected marginal benefit. Similarly, the condition for Pareto efficiency, with arbitrary weights on the welfare of boys and girls, is

$$c_B'(x_B^{**}) \times c_G'(x_G^{**}) = r. \tag{13}$$

We now examine a quasi-symmetric equilibrium in which all boys invest x_B^* and all girls invest x_G^*. Since only the top r fraction of boys will be matched, this corresponds to those having a realization of $\varepsilon \geq \tilde{\varepsilon}$, where $F(\tilde{\varepsilon}) = 1 - r$. In this case, a boy of type $\varepsilon \geq \tilde{\varepsilon}$ will be matched with a girl of type $\phi(\varepsilon, r)$, where

$$1 - F(\varepsilon) = r[1 - G(\phi(\varepsilon, r))].$$

The derivative of this matching function is given by

$$\phi_\varepsilon(\varepsilon, r) = \frac{f(\varepsilon)}{rg(\phi(\varepsilon, r))}.$$

That is, an increase in ε increases a boy's match quality relatively more quickly, since the distribution of girls is relatively thinner, since $r < 1$.

Those boys with realizations below $\tilde{\varepsilon}$ will not be matched and receive a payoff $\bar{u} < \underline{\eta}$, what we have called the misery effect. As Hajnal (1982) has noted, in Asian societies such as China and India, marriage rates have historically been extremely high (over 99 percent, as compared to the traditional "European marriage pattern" with marriage rates around 90 percent). Thus the misery effect is likely to be large in Asian societies.

The first-order condition for boys in an equilibrium in which all boys invest the same amount x_B^* while all girls invest the same amount x_G^* is given by

$$\frac{1}{r}\int_{\tilde{\varepsilon}(r)}^{\bar{\varepsilon}} \frac{f(\varepsilon)}{g(\phi(\varepsilon, r))} f(\varepsilon)\, d\varepsilon + f(\tilde{\varepsilon}(r))(\underline{\eta} + x_G^* - \bar{u}) = c_B'(x_B^*). \tag{14}$$

As compared to our previous analysis, we notice two differences. The first term is the improvement in match quality, and the sparseness of girls increases the investment incentives because of the term in $1/r$. Additionally, an increment in investment raises the probability of one's son getting matched, at a rate $f(\tilde{\varepsilon})$, and the marginal payoff equals the difference between matching with the worst-quality girl and receiving $\underline{\eta} + x_G^*$ and not being matched and receiving $\bar{u}$. An unbalanced sex ratio tends to amplify investments in boys, for two reasons. First, a given increment in investment pushes boys more quickly up the distribution of girls, and second, there is an incentive to invest in order to increase the probability of a match taking place at all, since there is a discontinuous payoff loss from not being matched at $\tilde{\varepsilon}$ due to the misery effect.

Similarly, the first-order condition for investment in girls is given by

$$r\int_{\underline{\eta}}^{\bar{\eta}} \frac{g(\eta)}{f(\phi^{-1}(\eta))} g(\eta)\, d\eta = c_G'(x_G^*). \tag{15}$$

Notice here that the role of $r < 1$ is to reduce investment incentives, since an increment in investment pushes a girl more slowly up the distribution of boy qualities. Furthermore, there is no counterpart to the misery effect for the scarcer sex, and the only reason to invest arises from the consequent improvement in match quality.

Since boys are in excess supply, a girl whose parents invest and whose quality realization is discretely lower than that of every other girl will still be able to find a partner. Such a girl will get a match payoff of $x_B^* + \tilde{\varepsilon}$, no matter how low her own quality. Thus, the conditions for the existence of a quasi-symmetric equilibrium are more stringent than in the balanced sex ratio case. Large downward deviations in investment will not be profitable provided that the dispersion in the qualities of boys is sufficiently large and the cost function for girls is sufficiently convex.

Proposition 2. If $r < 1$ and the noise is additive, there exists a unique quasi-symmetric equilibrium, provided that $f(\cdot)$ is sufficiently dispersed and $c_B(\cdot)$ is sufficiently convex. Investments are excessive relative to Pareto efficiency, for generic distributions of noise.

It is worth pointing out that even without the misery effect, there will be strictly excessive investments, even if the noise distributions are identical, unless they happen to be uniform. When $r < 1$, $g(\phi(\varepsilon))$ is not a linear transformation of $f(\varepsilon)$ unless f and g are uniform. Thus investment will be strictly greater than the efficient level.

As we have already noted, if F and G have the same distributions and $r = 1$, investments are utilitarian efficient. Thus a balanced sex ratio is sufficient to ensure efficiency of investments in this case. This provides an additional argument for the optimality of a balanced sex ratio, over and above the congestion externality identified in Bhaskar (2011).

We may use our model to evaluate the theoretical basis of the empirical work by Wei and Zhang (2011), attributing the high savings rate in China to the sex ratio imbalance. Given the condition for utilitarian efficiency $c'_B(x_B^{**}) = r$, investment in boys should actually fall as r decreases below one, from a utilitarian point of view. In a related signaling model, Hoppe et al. (2009) show that an increase in the number of men will increase total signaling by men; however, the effect on signaling by women is ambiguous and depends on the shape of the distribution of abilities among men. The shape of the distribution also matters for our model, but a further difficulty here is that investment by one side affects the incentives to invest by the other. In particular, the equilibrium choice of investment by girls enters the boys' first-order condition (14). This potentially would also make the investment by boys respond ambiguously to the sex ratio becoming less equal.

As an example, consider the case in which f and g are increasing and $f(\underline{\varepsilon}) = g(\underline{\eta}) = 0$:

$$r\int_{\underline{\eta}}^{\bar{\eta}} \frac{g(\eta)}{f(\phi^{-1}(\eta, r))} g(\eta)\, d\eta = \int_{\tilde{\varepsilon}(r)}^{\bar{\varepsilon}} g(\phi(\varepsilon, r))\, d\varepsilon.$$

Since g is increasing and both ϕ and the range of integration $[\tilde{\varepsilon}(r), \bar{\varepsilon}]$ are increasing in r, it follows that investment by girls is unambiguously increasing in r. Similarly,

$$\frac{1}{r}\int_{\tilde{\varepsilon}(r)}^{\bar{\varepsilon}} \frac{f(\varepsilon)}{g(\phi(\varepsilon, r))} f(\varepsilon)\, d\varepsilon = \int_{\underline{\eta}}^{\bar{\eta}} f(\phi^{-1}(\eta, r))\, d\eta.$$

Since f is assumed increasing and ϕ^{-1} is decreasing in r, overall this expression is decreasing in r. That is, the matching incentive for boys increases as the sex ratio becomes more uneven. However, the overall effect on boys' investments is ambiguous, as the left-hand side of (14) also depends on x_G^*, which is increasing in r. One can at least conclude that for r close to one, so that $\tilde{\varepsilon}$ is close to $\underline{\varepsilon}$ and $f(\tilde{\varepsilon})$ is close to zero, x_B^* is decreasing in r.

In summary, under some assumptions, an uneven sex ratio can indeed increase investment incentives for men but will also decrease incentives for women. The predicted effect on total investment is consequently ambiguous. An uneven gender ratio increases the relative weight of boys in the population and their increased investment may be enough to increase the total, but this is not guaranteed.

B. Talent Shocks and Complementarities with Investment

Consider next the case in which investment is in education and the uncertainty is talent risk. It is plausible that the return to education depends on the talent of the child. To model this, we suppose that quality is given by a multiplicative production function, $q^B(x, \varepsilon) = x\varepsilon$ and $q^G(x, \eta) = x\eta$, where ε and η are always strictly positive. Further, investment levels must be strictly positive since investments below the individually optimal level are strictly dominated. Investments in a quasi-symmetric equilibrium must therefore satisfy the first-order condition for equilibrium (8) and (9). Since $q_x^i = \varepsilon$ and $q_\varepsilon^i = x$, these can be rewritten as

$$\frac{x_G^*}{x_B^*}\int \frac{f(\varepsilon)}{g(\phi(\varepsilon))}\varepsilon f(\varepsilon)d\varepsilon = c_B'(x_B^*), \tag{16}$$

$$\frac{x_B^*}{x_G^*}\int \frac{g(\eta)}{f(\phi^{-1}(\eta))}\eta g(\eta)d\eta = c_G'(x_G^*). \tag{17}$$

Unlike the additive case, the "reaction function" for the boys is upward sloping in the girls' investments and vice versa. Intuitively, if quality is multiplicative, an increase in the girls' investment levels increases the dispersion in qualities on the girls' side, thereby increasing investment incentives for boys. Thus, with multiplicative shocks, one has interesting interaction effects between investments on the two sides of the market. First, we examine the implications of gender differences.

PROPOSITION 3. Assume multiplicative shocks that have the same mean for boys and girls and identical cost functions for the two sexes. If the distribution of shocks for boys is more dispersed than that for girls, that is, $F \geq_d G$, then there is underinvestment in boys and there is overinvestment in girls relative to the utilitarian efficient level.

Proof. From the first-order conditions, if $F \geq_d G$,

$$c_B'(x_B^*) = \frac{x_G^*}{x_B^*}\int \frac{f(\varepsilon)}{g(\phi(\varepsilon))}\varepsilon f(\varepsilon)d\varepsilon < \frac{x_G^*}{x_B^*}\mathbf{E}(\varepsilon),$$

$$c_G'(x_G^*) = \frac{x_B^*}{x_G^*}\int \frac{g(\eta)}{f(\phi^{-1}(\eta))}\eta g(\eta)\,d\eta > \frac{x_B^*}{x_G^*}\mathbf{E}(\eta).$$

Utilitarian investments x_B^{**} and x_G^{**} satisfy $c_B'(x_B^{**}) = \mathbf{E}(\varepsilon)$ and $c_G'(x_G^{**}) = \mathbf{E}(\eta)$. Since $\mathbf{E}(\varepsilon) = \mathbf{E}(\eta)$ and the cost functions are the same, $c_B'(x_B^*) < \mathbf{E}(\varepsilon)$ and $c_G'(x_G^*) > \mathbf{E}(\eta)$. QED

Multiplicative shocks have the interesting implication that an increase in investment by one side, say girls, also increases the dispersion in quality among girls, thereby providing more incentives to invest for the boys. So if shocks are relatively less variable among girls, the induced investments are such that the resulting difference in variability in quality between the two sides is less pronounced. Since girls invest more and boys invest less, this raises quality dispersion among girls and reduces it among boys.

Both the additive model and the multiplicative model show that the side with more dispersed shocks has weaker incentives for investment. A similar argument is also found in Hoppe et al. (2009), where differences in the dispersion of exogenously given unobserved qualities affect signaling expenditures by men and women. The additive shocks model yields conclusions similar to those of Hoppe et al. since investments by one side do not affect investment incentives on the other side. The multiplicative model is richer, since increased investments by girls raise the incentives to invest for boys.

These interaction effects have interesting implications also when the mean value of shocks differs between the sexes. Suppose that $F(\varepsilon) = G(\varepsilon + k)$ for some $k > 0$, so that average quality is higher among the girls but the distributions are of the same type. An example would be the case in which shocks are normally distributed, with the girls having a higher mean. Since $f(\varepsilon)/g(\phi(\varepsilon)) = 1$ for all values of ε, the first-order conditions reduce to

$$c_B'(x_B^*) = \frac{x_G^*}{x_B^*}\mathbf{E}(\varepsilon),$$

$$c_G'(x_G^*) = \frac{x_B^*}{x_G^*}\mathbf{E}(\eta).$$

From the first-order conditions, under the assumption that the genders have identical cost functions, we see that women invest more than men. However, the interaction effects imply that women invest less than the utilitarian efficient amount, and men invest more than the utilitarian level. Thus, in the multiplicative case, investment behavior partially offsets differences in mean quality.

Finally, the multiplicative model also shows us that facilitating investment by one side, say girls, also increases investment incentives for boys. In some developing countries such as India, governments have sought to overcome discrimination against girls by subsidizing their education. Increased investment in girls raises the dispersion in their quality, thereby providing greater incentives to invest for the boys.

1. Investment under Traditional Gender Roles

Suppose that shocks are additive for women but multiplicative for men. One example is a traditional society, where women do not work, and so investment in them takes the form of a dowry, while parents invest in their sons' human capital. This interpretation fits our model if investments toward dowries must take place in advance; that is, parents must forgo consumption in order to save for their daughter's dowry. The first-order condition for investments in boys is

$$\frac{1}{x_B^*}\int \frac{f(\varepsilon)}{g(\phi(\varepsilon))}\varepsilon f(\varepsilon)d\varepsilon = c_B'(x_B^*). \tag{18}$$

Notice that this is independent of the investment level of the girls, since the quality of girls is additive. On the other hand, since the dispersion of qualities among the boys increases with their investment level, the investment by girls is increasing in boys' investments, as the girls' first-order condition shows:

$$x_B^*\int \frac{g(\eta)}{f(\phi^{-1}(\eta))}g(\eta)d\eta = c_G'(x_G^*). \tag{19}$$

This mixed model can provide an explanation for why dowries may increase during the process of development, as, for example, in India.[16] Suppose that the marginal costs of investing in human capital fall, so that $c_B'(x_B)$ is reduced at any value of x_B. From (18), x_B^* must go up. From (19), x_G^* also increases. Intuitively, the increased investment by boys increases the variability in their quality, thereby increasing the incentives for the parents of girls for investing in their dowries. A similar argument can also be made if the return to human capital goes up; this will increase investment levels by boys, increasing the dispersion in their qualities. The increased dispersion in boys' qualities increases the investment incentives for girls.

2. Efficiency in a Generalized Multiplicative/Additive Model

We now examine whether investments are Pareto efficient in a generalized model in which quality has multiplicative as well as additive components. Assume that quality satisfies assumption 3, set out in Section III. Utilitarian efficiency requires

$$c_B'(x_B^{**}) = \theta_B + (1-\theta_B)\gamma'(x_B^{**})\mathbf{E}(\varepsilon)$$

[16] Anderson (2007) surveys the evidence on dowries, while Anderson (2003) provides an alternative explanation for the increase in dowries during modernization, based on social stratification.

and

$$c_G'(x_G^{**}) = \theta_G + (1 - \theta_G)\gamma'(x_G^{**})\mathbf{E}(\eta).$$

Pareto-efficient investments are such that

$$c_B'(x_B^{**}) \times c_G'(x_G^{**}) = [\theta_B + (1 - \theta_B)\gamma'(x_B^{**})\mathbf{E}(\varepsilon)] \times [\theta_G + (1 - \theta_G)\gamma'(x_G^{**})\mathbf{E}(\eta)].$$

The following theorem shows that there will be generic overinvestment relative to the above Pareto efficiency condition if the densities are both symmetric, as long as they differ.

THEOREM 3. Suppose that quality is generalized additive/multiplicative and that the distributions f and g are symmetric. Then in any quasi-symmetric equilibrium, investments are generically excessive relative to Pareto efficiency.

This overinvestment result applies both to the pure multiplicative model and to the one with traditional gender roles. The proof is based on the Cauchy-Schwarz inequality, and the result is robust. If f and g are symmetric and linearly independent, investment will be strictly too high. Now if we perturb the distributions so that $\tilde{f}$ is close to f and $\tilde{g}$ to g, then $c_B'(x_B^*) \times c_G'(x_G^*)$ will still be greater than those required for efficiency, since the integrals defining this are continuous in the distributions. In other words, we will have excessive investments even with asymmetric distributions as long as the asymmetries are not too large.

Why does the result require that the asymmetry not be too large? To provide some intuition for this, assume a purely multiplicative quality, and return to the example in which the distribution of shocks is uniform on $[0, 1]$ for men and the density function for women, $g(\eta) = 2\eta$, on $[0, 1]$. Here again, the ratio of the densities that is relevant for men, $f(\varepsilon)/g(\phi(\varepsilon))$, is relatively large when ε is low. While these values of ε still have large weight (since $f(\varepsilon)$ is constant in ε), in the multiplicative case, the payoff to investment is low when ε is small. Under symmetry, neither particularly low values nor particularly high values of ε have any special weight, and thus the inefficiency result applies.

IV. No Gender Difference Implies Efficiency

We now consider the implications of ex ante heterogeneity, where individuals differ even before shocks are realized, beginning with an illustrative example. Assume that the sex ratio is balanced. Suppose that we have two classes, H and L, with fractions θ_H and θ_L in the population. Assume that the marginal costs of investment are lower for the upper class, H. Let $c_H(\cdot)$ and $c_L(\cdot)$ be the cost functions, which depend on class but not on gender, where $c_H'(x) < c_L'(x)$ for any x. Let $f_i(\cdot)$ and $g_i(\cdot)$, $i \in \{H, L\}$, denote the density function of shocks for the boys from class i and the girls from class i, respectively. Assume that the quality function is additive in the shocks and investment.

Consider a profile of investments $(x_{HB}, x_{LB}, x_{HG}, x_{LG})$, where each individual who belongs to the same class and the same gender chooses the same investment. This profile induces a distribution of qualities for the boys, $\tilde{F}(q)$, and for girls, $\tilde{G}(p)$. Since any stable measure preserving matching $\tilde{\phi}$ must be assortative, we must have $\tilde{F}(q) = \tilde{G}(\tilde{\phi}(p))$.

Let x^*_{HB}, x^*_{LB}, x^*_{HG}, x^*_{LG} be the equilibrium investment levels. Suppose that the distribution of qualities in both the sexes has a connected support without any gaps. In class i, the first-order condition for investment in boys is given by

$$\int_{q_{\min}}^{q_{\max}} \frac{\tilde{f}(q)}{\tilde{g}(\phi(q))} f(q - x^*_{iB})\,dq = c'_B(x^*_{iB}). \tag{20}$$

The density function for boys' quality is given by (that for girls is analogous)

$$\tilde{f}(q) = \theta_H f_H(q - x^*_{HB}) + \theta_L f_L(q - x^*_{LB}).$$

Let the class differences be arbitrary, so that f_H can differ from f_L and g_H from g_L. However, assume that there are no gender differences, so that $f_H = g_H$ and $f_L = g_L$. Consider a *gender-neutral strategy* profile, where investments depend on class but not on gender, so that $x^*_{iB} = x^*_{iG}$ for $i \in \{H, L\}$. Since the shocks do not vary between the sexes, the induced distribution of qualities will be identical in the two sexes. That is, for any value q, $\tilde{F}(q) = \tilde{G}(q)$, implying that $\tilde{\phi}(q) = q$. This in turn implies that $\tilde{f}(q) = \tilde{g}(\phi(q))$. Therefore, the left-hand side of equation (20) equals one.

Consider a utilitarian social planner who puts equal weight on each type of individual, irrespective of gender or social class. Since the marginal benefit of additional investment in a boy is unity, for any girl who is matched with him, such a planner would set the marginal cost of investment to one. We conclude, therefore, that investment in boys is utilitarian efficient if there are no gender differences, even if there is large heterogeneity between classes. Similarly, investment in girls is utilitarian efficient.

This argument is very general: provided that there are no differences between the sexes, equilibrium investments will be utilitarian efficient even if there is wide heterogeneity within each sex. Assume that there is a finite set of types, indexed by $i \in \{1, 2, \ldots, n\}$. Type i has a measure μ^i of boys and an equal measure of girls, with $\sum_{i=1}^{n} \mu^i = 1$. A boy or girl of type i has an idiosyncratic component of quality, ε, that is distributed with a density function $f_i(\varepsilon)$ and cdf $F_i(\varepsilon)$. We shall assume a general quality function $q(x, \varepsilon)$, where q is continuous, increasing in both arguments, and differentiable and concave in x, the investment. We assume that the quality function is identical for the two sexes and is therefore not indexed by gender. The cost of investment may also depend on type and is denoted $c_i(x)$. We assume that assumption 1 holds for each type; that is, it holds for each cost function c_i and each density function f_i and the quality function.

We assume no gender difference: Men and women are symmetric with regard to costs of investment and the idiosyncratic component of quality. Specifically, for any type i, (i) there are equal measures of men and women, (ii) the investment cost functions and quality functions do not differ across the sexes, and (iii) the idiosyncratic component of quality has the same distribution, f_i, that depends on type but not on gender.

The assumption of no gender difference is strong, but there are reasonable conditions under which it is satisfied. Suppose that investment costs or the idiosyncratic component depend on the "type" of the parent (e.g., parental wealth, human capital, or social status) but not directly on gender. If the gender of the child is randomly assigned, with boys and girls having equal probability, then no gender difference will be satisfied.

A utilitarian efficient profile of investments $(x^{**}_{Bi}, x^{**}_{Gi})^n_{i=1}$ is one that maximizes the sum of payoffs of all individuals, irrespective of type or gender. This satisfies the conditions below for all i; that is, the marginal social benefit from increased quality must equal the marginal cost to the individual:

$$c_i'(x^{**}_{Bi}) = \int q_x(x^{**}_{Bi}, \varepsilon) f_i(\varepsilon)\, d\varepsilon, \tag{21}$$

$$c_i'(x^{**}_{Gi}) = \int q_x(x^{**}_{Gi}, \eta) g_i(\eta)\, d\eta. \tag{22}$$

Under assumption 1, the cost function c_i is strictly convex and $q_{xx}(\cdot) \leq 0$, so that the above conditions are sufficient for the profile to be utilitarian efficient. Assuming no gender difference, utilitarian efficiency requires that individuals of the same type choose the same investments even if they differ in gender, that is, $x^{**}_{Bi} = x^{**}_{Gi} \equiv x^{**}_i$ for all i.

Consider now a quasi-symmetric strategy profile $((x^*_{Bi})^n_{i=1}, (x^*_{Gi})^n_{i=1})$, specifying investment levels for each type of each gender. This profile, in conjunction with the realizations of idiosyncratic shocks, induces a cdf of qualities, $\tilde{F}$, in the population of boys. Since ε is assumed to be atomless and q is continuous, $\tilde{F}$ admits a density function $\tilde{f}$, although its support may not be connected if the investment levels of distinct types are sufficiently far apart (i.e., there may be gaps in the distribution of qualities). Similarly, let $\tilde{G}$ denote the cdf of girl qualities, given $(x_{Gi})^n_{i=1}$. A stable measure preserving matching must be assortative, so that a boy of type q is matched to a girl of type $\phi(q)$ if and only if $\tilde{F}(q) = \tilde{G}(\phi(q))$. Thus the distributions $\tilde{F}$ and $\tilde{G}$ define the match payoffs associated with equilibrium investments for each type of boy and each type of girl. For the profile $((x^*_{Bi})^n_{i=1}, (x^*_{Gi})^n_{i=1})$ to be an equilibrium, it must satisfy the first-order conditions for each type i for the boys and girls, respectively:

$$c'_i(x^*_{Bi}) = \int \tilde{\phi}'(\cdot)q_x(x^*_{Bi}, \varepsilon)f_i(\varepsilon)d\varepsilon, \tag{23}$$

$$c'_i(x^*_{Gi}) = \int (\tilde{\phi}^{-1})'(\cdot)q_x(x^*_{Gi}, \eta)g_i(\eta)d\eta. \tag{24}$$

We shall call a strategy profile *gender-neutral* if $x_{Bi} = x_{Gi}$ for all i, so each type of parent invests the same amount regardless of the gender of their child. Suppose that $((x^*_{Bi})^n_{i=1}, (x^*_{Gi})^n_{i=1})$ is gender-neutral and is an equilibrium. Under the assumption of no gender difference, the induced distributions of qualities are identical on the two sides, that is, $\tilde{F}(\cdot) = \tilde{G}(\cdot)$. Thus $\tilde{\phi}(\cdot)$ is the identity map on the support of $\tilde{F}(\cdot)$. The first-order conditions reduce to

$$c'_i(x^*_{Bi}) = \int q_x(x^*_{Bi}, \varepsilon)f_i(\varepsilon)d\varepsilon = c'_i(x^*_{Gi}). \tag{25}$$

The first-order conditions for an equilibrium that is gender-neutral, (25), coincide with the first-order conditions for utilitarian efficiency, (21) and (22). Thus if a gender-neutral equilibrium exists, it must be utilitarian efficient. Also, if large deviations from the utilitarian profile are unprofitable for every type, then it is the unique gender-neutral equilibrium.

A profile of investments, $((x_{Bi})^n_{i=1}, (x_{Gi})^n_{i=1}))$, has no quality gaps if the induced distributions of qualities, $\tilde{F}(q)$ and $\tilde{G}(p)$, have connected supports. Large deviations will be unprofitable as long as there are no quality gaps.

THEOREM 4. Suppose that there is no gender difference and assumption 1 is satisfied. The utilitarian efficient profile of investments is a gender-neutral equilibrium if it has no quality gaps and $\bar{u}$ is sufficiently small. A gender-neutral strategy profile is an equilibrium only if it is utilitarian efficient.

The efficiency result applies plausibly to a nonmarriage context. Consider a single-population matching model, where quality is a one-dimensional scalar variable. An example is partnership formation, for example, firms consisting of groups of lawyers. Theorem 4 implies that one has efficient ex ante investments, even without transferable utility. While the formal proof restricts attention to pairwise matching, the extension to matches consisting of more than two partners is immediate.

The intuition for the efficiency result is as follows. Consider a gender-neutral profile of investments, where there are no quality gaps. Then a boy of quality q will be matched with a girl of the same quality, that is, $\tilde{\phi}(q) = q$. Thus the marginal return to investment equals the increment to his own quality, and thus private incentives and utilitarian welfare are perfectly aligned.

This result does require the no quality gap assumption, which will be satisfied if the support of the shocks is large enough or if there is sufficient similarity between adjacent types so that their equilibrium investments are not too far apart. Interestingly, if there are quality gaps, then there is a tendency for overinvestment rather than underinvestment. Let us return to the two-class illustrative example at the beginning of this section and suppose that the differences in utilitarian investments between the rich and the poor are so large that there is a quality gap. Suppose that an individual boy deviates from this profile and has a quality realization that is greater than that of the best poor boy and smaller than that of the worst rich boy. Assume that the deviator is assigned either the match of the former or that of the latter, each with probability one-half. Under such a matching rule, the poor boys would have an incentive to deviate upward: the rich boys have no incentive to deviate downward.[17]

V. A Model with Finite Numbers

We now set out a model with finitely many boys and girls, where there is uncertainty as to whether there are slightly more boys than girls or the reverse.[18] Thus the lowest-quality boy (or girl) will be unmatched with probability one-half. We show that the payoffs in this finite model converge to those in the continuum model as the number of participants becomes large. This provides a justification for our assumption in the continuum case, that a downward-deviating agent, whose quality is below the support of the equilibrium distribution of qualities, is unmatched with probability one-half. We also show that if the number of agents is sufficiently large, there exists a unique quasi-symmetric equilibrium of the finite model, which converges to the equilibrium of the continuum model as the number of agents tends to infinity.

Assume that there are $2n + 1$ agents, with their sex being determined as follows: n of the agents are randomly chosen (equiprobably, so that

[17] The matching rule can be justified as the limit of a model with a large but finite number of agents of each type, along the lines of the argument in the next section. Efficiency can be obtained even with quality gaps if we modify the matching rule so that a deviator is always assigned the match of the next-worst individual. This matching rule is formally correct in the continuum model but does not seem to correspond to the limit of a reasonable finite model.

[18] We thank Roger Myerson for suggesting this approach. See also Myerson (1998) for large games with a random set of players.

each agent has an equal chance of being chosen), and then a fair coin is tossed to determine whether these n chosen individuals are all male or all female. The $n + 1$ unchosen individuals are then specified to be of the opposite sex. We assume that at the time of investment, individuals each know their sex but not whether they were among the n chosen individuals. Thus, at the investment stage, an individual does not know whether boys or girls are in excess.

For tractability, we assume that quality is additive in investment and in the idiosyncratic shock. Shocks are i.i.d. and are drawn from F for the boys and G for the girls. Agents are matched assortatively in terms of quality, and so all agents are matched except the lowest-quality agent on the long side of the market. Suppose that all girls invest x_G and all boys invest x_B. If a single boy deviates and invests $x_B + \Delta$ and his shock value is ε, then since his quality equals $\varepsilon + \Delta + x_B$, it is the same as if he had invested x_B and had shock value $\varepsilon + \Delta$. Consequently, his prospects for marriage are the same as if he did not deviate (and hence are independent of x_B) and had a shock value $\varepsilon + \Delta$, with the caveat that if $\varepsilon + \Delta > \bar{\varepsilon}$, he marries the highest-quality girl, and if $\varepsilon + \Delta < \underline{\varepsilon}$, he marries the lowest-quality girl with probability $n/(2n + 1)$ and is unmarried with probability $(n + 1)/(2n + 1)$. The shock received by the girl that this deviating boy is matched with is defined to be $\bar{u} - x_G$ if the boy is unmatched. Let $\phi_n(\varepsilon + \Delta)$ denote the expected shock value of the boy's match.

Then the expected quality of the girl that the boy is matched with equals $\phi_n(\varepsilon + \Delta) + x_G$. Thus the ex ante expected benefit of a boy, averaged over his shock realizations, when he invests $x_B + \Delta$ equals

$$B_n(\Delta, x_G) = \int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \phi_n(\varepsilon + \Delta) f(\varepsilon)\, d\varepsilon + x_G. \tag{26}$$

The benefit function $B_n(\Delta, x_G)$ is differentiable with respect to Δ, and its derivative at $\Delta = 0$ is strictly positive. Since marginal investment costs are strictly negative at zero and unbounded as investments tend to the upper bound, any quasi-symmetric equilibrium (x^*_{Bn}, x^*_{Gn}) must lie in the interior of the set of feasible investments. Thus equilibrium investments must satisfy the first-order conditions:

$$c'_B(x^*_{Bn}) = B'_n(0, x^*_{Gn}),$$

$$c'_G(x^*_{Gn}) = B'_n(0, x^*_{Bn}).$$

We have the following result.

THEOREM 5. Suppose that quality is additive and that assumptions 1 and 2 are satisfied. For n sufficiently large, the game with uncertain finite numbers has a unique quasi-symmetric equilibrium with investments (x^*_{Bn}, x^*_{Gn}), provided that $\bar{u}$ is sufficiently small. Further, $\lim_{n\to\infty}(x^*_{Bn}, x^*_{Gn}) = (x^*_B, x^*_G)$, the equilibrium investments in the continuum model.

We prove this result in the Appendix by showing that the benefit function in the finite model and its derivatives converge to their counterparts in the continuum model. Since theorem 2 shows that one has generic overinvestment in the continuum case, our main convergence result here, theorem 5, implies that there is excessive investment when the number of agents is sufficiently large. However, exactly how investment incentives depend on population size is quite complex, and we do not analyze it here.

VI. Conclusions

We examined a model of marriage with investments that have stochastic returns. This approach ensures the existence of a unique pure strategy equilibrium, in an area where models often have multiple equilibria or equilibria only in mixed strategies. Our main result is that investments are inefficiently high, generically. The intuition for our inefficiency result is somewhat subtle: it is not due to the usual positional externality that arises in tournaments, since investments in our context are not wasteful. Investments by boys increase welfare for the girls and vice versa. Indeed, when the two sides or sexes are identical and there are no gender differences, one gets efficient investments. However, when there are differences between the sexes, there are some realizations of shocks in which boys have a relatively higher incentive to invest and other realizations in which girls have a relatively higher incentive to invest. Each sex gives greater weight to those states in which they have relatively greater incentives, giving rise to overinvestment. When the sexes are identical, at every shock realization, both sexes have identical investment incentives, which ensures efficiency. Our model also has interesting observational implications. For example, if shocks are more variable for boys as compared to girls, boys invest less than girls. If there is an unbalanced sex ratio, the abundant sex invests more, while the scarcer one invests less.

While formal tests of the Pareto efficiency of investments at the aggregate level are yet to be developed, Wei and Zhang (2011) present evidence that the sex ratio imbalance drives higher savings in China by the parents of boys, which is possibly inefficiently high. Similarly, the global boom in the higher education of women is arguably not explained by higher returns to education on the labor market (see Becker, Hubbard, and Murphy 2010). Our model suggests that matching considerations from the marriage market could explain the greater investment incentives of girls.

Appendix

Proofs

Proof of Theorem 1

We first derive the first-order conditions as given in (8) and (9). We then show that these first-order conditions are sufficient and that no individual can benefit from large deviations. Next, we show that the first-order conditions define a unique symmetric best response for the boys as a function of the investment level of the girls and vice versa. These best-response functions are continuous, establishing existence of equilibrium. Finally, we show that under assumption 3 equilibrium is unique.

Note that any investment $x < \bar{x}_i$ is strictly dominated, since $c_i'(x) < 0$ if $x < \bar{x}_i$ and quality is increasing in x, implying that the marginal benefit on the marriage market is positive. So we may restrict attention to profiles in which every agent invests strictly positive amounts. Consider a quasi-symmetric equilibrium in which all boys invest $x_B > 0$ and all girls invest $x_G > 0$. Suppose that a parent of a boy deviates from this equilibrium and invests $x_B + \Delta$ in his son. Without loss of generality, we may restrict Δ to lie in a compact interval $[\underline{\Delta}, \bar{\Delta}]$, where $\underline{\Delta} < 0 < \bar{\Delta}$. Let Δ' be defined by $q^B(x_B + \Delta', \bar{\varepsilon}) = q^B(x_B, \underline{\varepsilon})$ if there exists $x_B + \Delta' \geq 0$ that solves this equation; otherwise, let $\Delta' = -x_B$. Define $\underline{\Delta} \coloneqq \max\{\Delta', \bar{x}_B - x_B\}$. This definition reflects two facts. First, deviations below $\bar{x}_B$ are unprofitable. Second, very large downward deviations cause a boy to be below the support of the equilibrium distribution of boy qualities and thus unmatched with probability one-half; we assume that $\bar{u}$ is sufficiently small that such deviations are unprofitable.[19] Let Δ'' be defined by $q^B(x_B + \Delta'', \underline{\varepsilon}) = q^B(x_B, \bar{\varepsilon})$, and define $\bar{\Delta} \coloneqq \min\{\Delta'', \tilde{x}_B - x_B\}$, where $\tilde{x}_B$ is the upper bound on investments introduced in assumption 1. The definition of Δ'' reflects the fact that there is no advantage in deviating more than Δ'', since there is no better outcome than being matched with the best-quality woman.

Suppose that a parent of a boy deviates from this equilibrium and invests $x_B + \Delta$ in his son, where $\Delta \in [\underline{\Delta}, \bar{\Delta}]$. If the realization of the shock for his son is ε, the son will hold the same rank in the population of boys as a boy with a shock level ξ, where $\xi(x, \varepsilon, \Delta)$ is defined by the equation

$$q^B(x_B + \Delta, \varepsilon) = q^B(x_B, \xi(x_B, \Delta, \varepsilon)).$$

For example, in the additive case, $\xi(x_B, \Delta, \varepsilon) = \varepsilon + \Delta$. Given this deviation, the boy now holds rank $F(\xi)$ in the population of boys and can expect a match with a girl holding rank $G(\phi(\xi))$ in the population of girls. She would be of quality $q^G(x_G, \phi(\xi))$. Applying the implicit function theorem to the above equation, it is easy to show that $\xi(x_B, \Delta, \varepsilon)$ has the following properties:

[19] If $\Delta < \Delta'$, then $B(\Delta) = [q^G(x_G, \eta) + \bar{u}]/2$. Thus the overall payoff from such a deviation is no greater than $[q^G(x_G, \eta) + \bar{u}]/2 - c_B(\bar{x}_B)$. Since the equilibrium match payoff of a boy is no less than $q^G(x_G, \underline{\eta})$, if $\bar{u} < q^G(x_G, \underline{\eta}) - 2[c_B(x_B) - c_B(\bar{x}_B)]$, such a deviation is unprofitable.

$$\xi_\Delta > 0, \xi_\varepsilon > 0, \xi_{\varepsilon\varepsilon} = 0, \xi_{\Delta\varepsilon} \geq 0, \xi_{\Delta x} \leq 0, \xi_{\Delta\Delta} \leq 0, \tag{A1}$$

given our assumptions that $q^B_{xx} \leq 0$, $q^B_{\varepsilon x} \geq 0$, and $q^B_{\varepsilon\varepsilon} = 0$.

Let $\Delta \in (0, \bar{\Delta}]$ and define $\hat{\varepsilon}(\Delta)$ by $\xi(x_B, \Delta, \hat{\varepsilon}) = \bar{\varepsilon}$. That is, given an upward deviation of Δ, $\hat{\varepsilon}(\Delta)$ is the shock value that results in the same quality as the highest-ranked nondeviating boy, and given our assumption that $\Delta \leq \bar{\Delta}$, $\hat{\varepsilon}(\Delta) \geq \underline{\varepsilon}$. If a deviating boy has a shock value that is greater than $\hat{\varepsilon}(\Delta)$, then he matches for sure with the highest-ranking girl, with quality $q^G(x_G, \bar{\eta})$, where $\bar{\eta} = \phi(\bar{\varepsilon})$. Thus, if all other boys invest an amount x_B and all girls x_G, then the expected match quality or benefit $B(\Delta)$ of a boy investing $x_B + \Delta$, where $\Delta > 0$, is given by[20]

$$B(\Delta) = \int_{\underline{\varepsilon}}^{\hat{\varepsilon}(\Delta)} q^G(x_G, \phi(\xi(x_B, \Delta, \varepsilon)))f(\varepsilon)\, d\varepsilon + [1 - F(\hat{\varepsilon}(\Delta))]q^G(x_G, \bar{\eta}). \tag{A2}$$

Evaluating $B'(0)$ is troublesome because downward deviations induce a positive probability of being unmatched, while upward deviations do not. Thus we will first consider upward deviations to derive the limit $\lim_{\Delta \to 0^+} B'(\Delta)$, which we denote by $B'(0^+)$. Then we consider downward deviations to derive $\lim_{\Delta \to 0^-} B'(\Delta)$, which we denote by $B'(0^-)$. The two turn out to be equal, from which we conclude (from the mean value theorem) that $B'(0^+) = B'(0^-) = B'(0)$.

The derivative of the expected match or benefit with respect to Δ when $\Delta \in (0, \bar{\Delta})$ equals

$$B'(\Delta) = \int_{\underline{\varepsilon}}^{\hat{\varepsilon}(\Delta)} q^G_\eta(x_G, \phi(\xi(x_B, \Delta, \varepsilon)))\phi'(\xi(x_B, \Delta, \varepsilon))\xi_\Delta(x_B, \Delta, \varepsilon)f(\varepsilon)\, d\varepsilon. \tag{A3}$$

To take its limit as $\Delta \downarrow 0$, note that ϕ' and ξ_Δ are both continuous in Δ, and at $\Delta = 0$, they are given by

$$\phi'(\varepsilon) = \frac{f(\varepsilon)}{g(\phi(\varepsilon))}, \quad \xi_\Delta(x_B, \Delta, \varepsilon) = \frac{q^B_x(x_B, \varepsilon)}{q^B_\varepsilon(x_B, \varepsilon)}.$$

Similarly, $\xi(x_B, \Delta, \varepsilon)$ is continuous in Δ and equals ε at $\Delta = 0$, and so $\hat{\varepsilon} = \bar{\varepsilon}$ when $\Delta = 0$. Thus, we have

$$B'(0^+) = \int_{\underline{\varepsilon}}^{\bar{\varepsilon}} q^G_\eta(x_G, \phi(\varepsilon))\, \frac{f(\varepsilon)}{g(\phi(\varepsilon))}\, \frac{q_x(x_B, \varepsilon)}{q^B_\varepsilon(x_B, \varepsilon)}\, f(\varepsilon) d\varepsilon. \tag{A4}$$

Thus the right-hand derivative, $B'(0^+)$, equals the left-hand side of the first-order condition (8).

Turning to downward deviations, $\Delta \in (\underline{\Delta}, 0)$, define $\tilde{\varepsilon}(\Delta)$ by the relation $\xi(x_B, \Delta, \tilde{\varepsilon}) = \underline{\varepsilon}$; since $\Delta \geq \underline{\Delta}$, this ensures that $\tilde{\varepsilon}(\Delta) \leq \bar{\varepsilon}$. For $\Delta \in (\underline{\Delta}, 0)$, $B(\Delta)$ is given by

[20] Note that B is properly a function of (Δ, x_B, x_G); but from the point of view of an individual boy, x_B and x_G are fixed, and thus we write $B(\Delta)$ for simplicity.

$$B(\Delta) = \int_{\tilde{\varepsilon}(\Delta)}^{\bar{\varepsilon}} q^G(x_G, \phi(\xi(x_B, \Delta, \varepsilon)))f(\varepsilon)d\varepsilon + F(\tilde{\varepsilon}(\Delta))\frac{\bar{u} + q^G(x_G, \underline{\eta})}{2}. \tag{A5}$$

The final term depends on the following crucial assumption. If the deviating boy has a low shock realization, in the interval $[\underline{\varepsilon}, \tilde{\varepsilon}]$, his quality will be less than the lowest quality in the equilibrium distribution of boys' qualities. With equal probability, he will be left unmatched and have utility $\bar{u}$ or match with the lowest-quality girl. The derivative with respect to Δ, given $\Delta \in (\underline{\Delta}, 0)$, is

$$\begin{aligned} B'(\Delta) = &\int_{\tilde{\varepsilon}(\Delta)}^{\bar{\varepsilon}} q^G_\eta(x_G, \phi(\xi(\cdot)))\phi'(\xi(\cdot))\xi_\Delta(x_B, \Delta, \varepsilon)f(\varepsilon)d\varepsilon \\ &+ f(\tilde{\varepsilon}(\Delta))\frac{d\tilde{\varepsilon}}{d\Delta}\frac{\bar{u} - q^G(x_G, \underline{\eta})}{2}. \end{aligned} \tag{A6}$$

Since $\tilde{\varepsilon}$ is defined by the relation $q^B(x_B + \Delta, \tilde{\varepsilon}) = q^B(x_B, \underline{\varepsilon})$, $\tilde{\varepsilon} \to \underline{\varepsilon}$ as $\Delta \to 0$. Since $f(\underline{\varepsilon})$ is zero by assumption 1, the above derivative approaches (A4) as Δ goes to zero. Thus, the left and right derivatives exist and are equal at $\Delta = 0$ and give the first-order condition (8). The first-order condition for girls, (9), can similarly be derived.[21]

We now show that the integral defining $B'(\Delta)$ is well defined (even though $1/g(\phi(\varepsilon))$ is unbounded). As in the proof of theorem 2, we make a change of variables, from ε to η:

$$B'(0) = \int_{\underline{\eta}}^{\bar{\eta}} q^G_\eta(x_G, \eta)f(\phi^{-1}(\eta))\frac{q_x(x_B, \phi^{-1}(\eta))}{q^B_\varepsilon(x_B, \phi^{-1}(\eta))}d\eta.$$

Since $f(\phi^{-1}(\eta))$ is bounded, as is q_η, and since $q^B_\varepsilon(x_B, \phi^{-1}(\eta)) > 0$ and is constant as a function of $\phi^{-1}(\eta)$, the integral defining $B'(0)$ is well defined.

Assume for now that for any $x_G \in [0, \tilde{x}_G]$, there exists $\hat{x}_B(x_G)$ such that the boys' first-order condition (8) is satisfied (we demonstrate below that this is indeed the case). We show first that large deviations are not profitable, considering in turn upward deviations and downward deviations.

Upward deviations.—Let $B''(\Delta)$ denote the derivative of $B'(\Delta)$ on $(0, \bar{\Delta})$.[22] Let $U''_B(\Delta, \hat{x}_B) = B''(\Delta) - c''_B(\hat{x}_B + \Delta)$. We show that $U''_B(\Delta, \hat{x}_B) < 0$ for $\Delta \in (0, \bar{\Delta})$, which implies $B'(\Delta) - c'_B(\hat{x}_B + \Delta) < B'(0) - c'_B(\hat{x}_B)$, so that no upward deviation is profitable. Suppressing most arguments, $B''(\Delta)$ on $(0, \bar{\Delta})$ can be written as

[21] In some of our examples, e.g., those with a uniform distribution, the density f is nonzero at $\underline{\varepsilon}$, the lower bound of its support, and the left-hand derivative of the benefit function, (A4), is strictly greater than the right-hand derivative, (A6). Since optimality requires that the left-hand derivative of benefits is greater than or equal to marginal costs and that the right-hand derivative is less than or equal, there is a continuum of equilibria in this case. We focus in these examples on the equilibria with the smallest investments, i.e., where the right-hand derivative equals marginal costs. Given that our results demonstrate overinvestment, investments will be greater only in any other equilibrium.

[22] Note that $B'(\Delta)$ is not differentiable at zero.

$$B''(\Delta) = \int_{\underline{\varepsilon}}^{\hat{\varepsilon}(\Delta)} [q^G_{\eta\eta}(\phi'\xi_\Delta)^2 + q^G_\eta \phi'' \xi^2_\Delta + q^G_\eta \phi' \xi_{\Delta\Delta}] f(\varepsilon) d\varepsilon + \frac{d\hat{\varepsilon}}{d\Delta} q^G_\eta \phi' \xi_\Delta f(\hat{\varepsilon}). \tag{A7}$$

The second term is negative since $d\hat{\varepsilon}/d\Delta < 0$ and q^G_η, ϕ', and ξ_Δ are all positive. Thus $B''(\Delta) < 0$ if the first term, under the integral sign, is negative, and this is the case if all three functions $q^G_{\eta\eta}$, ϕ'', and $\xi_{\Delta\Delta}$ are (weakly) negative.[23] First, $q^G_{\eta\eta} = 0$ by assumption 1. Second, we have $\xi_{\Delta\Delta} \le 0$ from (A1). This leaves $\phi''(\cdot)$.

Invoking assumption 2, either (*a*) F and G are of the same type or (*b*) $f(\cdot)$ and $g(\cdot)$ are increasing. If point *a* is true, then $G(x) = F(a + bx)$, which implies that $\phi(\varepsilon)$ is linear and $\phi''(\cdot) = 0$. Thus, $B''(\Delta) < 0$.

We now consider point *b*: $f'(\varepsilon) \ge 0$ and $g'(\eta) \ge 0$. The expression (A7) can be written as

$$B''(\Delta) = \int_{\underline{\varepsilon}}^{\hat{\varepsilon}(\Delta)} q^G_\eta \phi' \xi_{\Delta\Delta} f(\varepsilon) d\varepsilon + \int_{\underline{\varepsilon}}^{\hat{\varepsilon}(\Delta)} q^G_\eta \phi'' \frac{\xi^2_\Delta}{\xi_\varepsilon} \xi_\varepsilon f(\varepsilon) d\varepsilon - q^G_\eta \phi' \frac{\xi^2_\Delta}{\xi_\varepsilon} f(\hat{\varepsilon}), \tag{A8}$$

given $q^G_{\eta\eta} = 0$ by assumption and since $d\hat{\varepsilon}/\Delta = -\xi_\Delta/\xi_\varepsilon$ when $\hat{\varepsilon}(\Delta) > \underline{\varepsilon}$. Note that the first integral is nonpositive, as a result of our finding that $\xi_{\Delta\Delta} \le 0$. We now show that the sum of the second and third terms is negative. Starting with the second term, note that the ratio $\xi^2_\Delta/\xi_\varepsilon$ is increasing in ε because $q_{x\varepsilon} \ge 0$ by assumption, $f(\varepsilon)$ is also increasing by assumption, and their product is also increasing. Then, by the second mean value theorem for integrals,[24] there is a $c \in [\underline{\varepsilon}, \hat{\varepsilon}]$ such that the second integral is equal to

$$f(\hat{\varepsilon}) \frac{\xi^2_\Delta(\hat{\varepsilon})}{\xi_\varepsilon(\hat{\varepsilon})} \int_c^{\hat{\varepsilon}(\Delta)} q^G_\eta \phi'' \xi_\varepsilon(\varepsilon) d\varepsilon = f(\hat{\varepsilon}) \frac{\xi^2_\Delta(\hat{\varepsilon})}{\xi_\varepsilon(\hat{\varepsilon})} [q^G_\eta(x_G, \phi(\bar{\varepsilon}))\phi'(\bar{\varepsilon}) - q^G_\eta(x_G, \phi(\xi(c)))\phi'(\xi(c))],$$

where $\xi(c) = \xi(x_B, \Delta, c)$ and the arguments of ξ_Δ and ξ_ε are similarly abbreviated. That is, we can write the second and third terms of (A8) together as

$$-q^G_\eta(x_G x^*_G, \phi(c))\phi'(c) f(c) \frac{\xi^2_\Delta(\hat{x}_B, \Delta, c)}{\xi_\varepsilon(\hat{x}_B, \Delta, c)} < 0.$$

Thus $B''(\Delta) < 0$ on $(0, \bar{\Delta})$ if f is increasing.

Downward deviations.—Let $\Delta \in (\underline{\Delta}, 0)$. Differentiating (A6), one obtains

[23] The integral defining the function $B''(\Delta)$ can be shown to be well defined using an argument similar to that employed for $B'(\Delta)$.

[24] More specifically, this is the special case known as Bonnet's theorem, which considers the integral of the product of two functions, where one (here $f(\varepsilon)\xi^2_\Delta/\xi_\varepsilon$) is nonnegative and increasing (see, e.g., Bartle 2001, 194).

$$B''(\Delta) = \int_{\tilde{\varepsilon}(\Delta)}^{\bar{\varepsilon}} [q^G_{\eta\eta}(\phi'\xi_\Delta)^2 + q^G_\eta \phi'' \xi^2_\Delta + q^G_\eta \phi' \xi_{\Delta\Delta}] f(\varepsilon) d\varepsilon$$
$$- \frac{d\tilde{\varepsilon}}{d\Delta} q^G_\eta \phi' \xi_\Delta f(\tilde{\varepsilon}(\Delta)) + \left[f'(\tilde{\varepsilon}(\Delta)) \left(\frac{d\tilde{\varepsilon}}{d\Delta} \right)^2 + f(\tilde{\varepsilon}(\Delta)) \frac{d^2\tilde{\varepsilon}}{d\Delta^2} \right] \tag{A9}$$
$$\times \frac{\bar{u} - {}_q G^q(x_G, \underline{\eta})}{2}.$$

The limit as $\Delta \uparrow 0$, $B''(0^-)$, is given by

$$B''(0^-) = \int_{\underline{\varepsilon}}^{\bar{\varepsilon}} [q^G_{\eta\eta}(\phi'\xi_\Delta)^2 + q^G_\eta \phi'' \xi^2_\Delta + q^G_\eta \phi' \xi_{\Delta\Delta}] f(\varepsilon) d\varepsilon$$
$$- f'(\underline{\varepsilon}) \left(\frac{d\tilde{\varepsilon}}{d\Delta} \right)^2 \frac{q^G(x_G, \underline{\eta}) - \bar{u}}{2}. \tag{A10}$$

Since $f'(\underline{\varepsilon}) > 0$ by assumption 1, we can choose $\bar{u}$ sufficiently small so that $B''(0^-) < 0$. Since $B''(\Delta)$ is continuous to the left at 0, there exists $d > 0$ such that $B''(\Delta) < 0$ if $\Delta \in (-d, 0)$.

Let $d' \in (0, d)$. We now consider deviations in the set $(\underline{\Delta}, -d']$ and show that $B'(\Delta) \geq B'(0)$ if $\bar{u}$ is sufficiently small. One has from (A6)

$$B'(\Delta) - B'(0) = \int_{\tilde{\varepsilon}(\Delta)}^{\bar{\varepsilon}} q^G_\eta(x_G, \phi(\xi(\cdot)))\phi'(\xi(\cdot))\xi_\Delta(x_B, \Delta, \varepsilon) f(\varepsilon)\, d\varepsilon$$
$$+ f(\tilde{\varepsilon}(\Delta)) \frac{d\tilde{\varepsilon}}{d\Delta} \frac{\bar{u} - q^G(x_G, \underline{\eta})}{2}$$
$$- \int_{\underline{\varepsilon}}^{\bar{\varepsilon}} q^G_\eta(x_G, \phi(\varepsilon)) \frac{f(\varepsilon)}{g(\phi(\varepsilon))} \frac{q^B_x(x_B, \varepsilon)}{q^B_\varepsilon(x_B, \varepsilon)} f(\varepsilon)\, d\varepsilon.$$

The first term is positive for any Δ and is thus bounded below by zero. The function $f(\tilde{\varepsilon}(\Delta))$ is bounded away from zero given assumption 1 and given $\Delta \in (\underline{\Delta}, -d']$, which implies $\tilde{\varepsilon}(\Delta) \in [\bar{\varepsilon}, \tilde{\varepsilon}(-d')]$. Then

$$\frac{d\tilde{\varepsilon}}{d\Delta} = -\frac{q^B_\varepsilon(\hat{x}_B + \Delta, \varepsilon)}{q^B_x(\hat{x}_B + \Delta, \varepsilon)},$$

and given our assumptions $q^B_{x\varepsilon} \geq 0$ and $q^B_{xx} \leq 0$, and since $x_B + \underline{\Delta} \geq \bar{x}_B$, this is bounded above by $-[q^B_\varepsilon(\bar{x}_B, \varepsilon)/q^B_x(\bar{x}_B, \varepsilon)] < 0$. Thus, we can find $\bar{u}$ sufficiently small so that the above expression is always positive for $\Delta \in (\underline{\Delta}, -d']$.

We have therefore established that $B'(\Delta) \geq B'(0)$ for any $\Delta \in (\underline{\Delta}, -d']$ and that $B''(\Delta) < 0$ for any $\Delta \in [-d, 0)$, where $d > d'$. Since the cost function is strictly convex, this establishes that no downward deviation in the set $(\underline{\Delta}, 0)$ is profitable.

We now show that the first-order conditions define reaction functions. For any $x_G \in [0, \tilde{x}_G]$, let $\hat{x}_B$ be a solution to the first-order condition for the boys, that is,

$$\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} q_{\eta}^{G}(x_G, \phi(\varepsilon))\frac{f(\varepsilon)}{g(\phi(\varepsilon))}\,\frac{q_x^B(\hat{x}_B,\varepsilon)}{q_\varepsilon^B(\hat{x}_B,\varepsilon)} f(\varepsilon)d\varepsilon - c_B'(\hat{x}_B) = 0. \tag{A11}$$

Let $h(x_B, x_G)$ denote the left-hand side of equation (A11), evaluated at an arbitrary x_B and x_G. Since $q_{\eta x}^G \geq 0$, $\partial h/\partial x_G \geq 0$. Since $q_{xx}^B \leq 0$, $q_{\varepsilon x}^B \geq 0$, and $c_B''(x_B) > 0$, $h(\cdot)$ is strictly decreasing in x_B. Thus $\hat{x}_B$ is weakly increasing in x_G. The function $h(x_B, x_G)$ is strictly positive when $x_B = 0$ since $c_B'(0) < 0$. Since $c_B'(x) \to \infty$ as $x \to \tilde{x}_B$ and since the first term on the left-hand side of equation (A11) is bounded, $h(x_B, x_G)$ is negative for x_B sufficiently close to $\tilde{x}_B$. Thus, by the intermediate value theorem, there exists $\hat{x}_B(x_G) \in (0, \tilde{x}_B)$ such that $h(x_B, x_G) = 0$. Since $h(\cdot)$ is strictly decreasing in x_B, this value is unique. By the implicit function theorem, the reaction function for the boys, $\hat{x}_B(x_G)$, is differentiable (and thus continuous). An identical argument establishes that the reaction function for the girls, $\hat{x}_G(x_B)$, is differentiable and increasing.

We now show that there exists a profile such that the reaction functions cross. Let $\zeta : [0, \tilde{x}_B] \to [0, \tilde{x}_B]$ be defined by $\zeta(x) = \hat{x}_B(\hat{x}_G(x))$. Note that if x_B^* is a fixed point of ζ, then the profile $(x_B^*, \hat{x}_G(x_B^*))$ is such that the first-order condition is satisfied for boys and for girls. Since $\hat{x}_G(\cdot)$ and $\hat{x}_B(\cdot)$ are continuous functions, so is ζ. Note that $\zeta(0) \geq \bar{x}_B > 0$, since $\hat{x}_B(x) \geq \bar{x}_B$ for all x. Also, $\zeta(\tilde{x}_B) < \tilde{x}_B$, since $\hat{x}_B(x_G) < \tilde{x}_B$ for any x_G. Thus, the intermediate value theorem implies that ζ has a fixed point in $[0, \tilde{x}_B)$.

We now show uniqueness under assumption 3. Since investments below the individually optimal investments are dominated, any equilibrium (x_B^*, x_G^*) must satisfy $x_B^* > 0$ and $x_G^* > 0$. Since $q_\eta^G(x_G^*, \eta) > 0$ when $x_G^* > 0$, the left-hand side of the first-order condition for boys, (8), is strictly positive, and thus $c_B'(x_B^*) > 0$. Similarly, $c_G'(x_G^*) > 0$. Use assumption 3 to rewrite the equilibrium first-order condition for boys, (8), as

$$\begin{aligned} &\frac{\theta_G + (1-\theta_G)\gamma(x_G^*)}{\theta_B + (1-\theta_B)\gamma(x_B^*)}\int_{\underline{\varepsilon}}^{\bar{\varepsilon}}[\theta_B + (1-\theta_B)\gamma'(x_B^*)\varepsilon] \\ &\qquad \times \frac{f(\varepsilon)}{g(\phi(\varepsilon))} f(\varepsilon)d\varepsilon = c_B'(x_B^*). \end{aligned} \tag{A12}$$

So if we take the product of the two first-order conditions,

$$\begin{aligned} c_B'(x_B^*)c_G'(x_G^*) = &\int [\theta_G + (1-\theta_G)\gamma'(x_G^*)\eta]\frac{[g(\eta)]^2}{f(\phi^{-1}(\eta))}\,d\eta \\ &\times \int [\theta_B + (1-\theta_B)\gamma'(x_B^*)\varepsilon]\,\frac{[f(\varepsilon)]^2}{g(\phi(\varepsilon))}\,d\varepsilon > 0. \end{aligned} \tag{A13}$$

If we have two distinct equilibria, both x_B^* and x_G^* must be higher in one of the equilibria since the reaction functions are increasing. The product of the marginal benefits, that is, the right-hand side of (A13), is decreasing in x_B^* and x_G^* by the concavity of γ. The product of the marginal costs $c_B'(x_B^*)c_G'(x_G^*)$ is increasing because of the convexity of costs. So both equilibria cannot satisfy (A13) and we have a contradiction. We therefore have proved the existence and uniqueness of the quasi-symmetric equilibrium. QED

Proof of the Lognormal Example

Let quality be additive in x_B and ε for the boys and in x_G and η for the girls. Let $h(\varepsilon)$ and $k(\eta)$ be strictly increasing differentiable functions, such that $h(\varepsilon) \sim N(\mu_\varepsilon, \sigma_\varepsilon)$ and $k(\eta) \sim N(\mu_\eta, \sigma_\eta)$. Let $\tilde{f}$ and $\tilde{g}$ denote the density functions of the shocks, and let f denote the density function of $h(\varepsilon)$ and g denote the density of $k(\eta)$. The matching $\phi(\varepsilon)$ is defined by

$$\tilde{F}(\varepsilon) = \tilde{G}(\phi(\varepsilon)) \Leftrightarrow F(h(\varepsilon)) = G(k(\phi(\varepsilon))).$$

This implies that

$$\phi'(\varepsilon) = \frac{f(h(\varepsilon))h'(\varepsilon)}{g(k(\phi(\varepsilon))k'(\phi(\varepsilon))}.$$

The first-order condition for boys is given by

$$\int_0^\infty \phi'(\varepsilon)\tilde{f}(\varepsilon)d\varepsilon = c_B'(x_B^*).$$

Let us now make a change of variables, from ε to h. Since $\tilde{F}(\varepsilon) = F(h(\varepsilon))$, we also have that $\tilde{f}(\varepsilon) = f(h(\varepsilon)) \cdot h'(\varepsilon)$. Furthermore, $dh = h'(\varepsilon)d\varepsilon$. Thus

$$B'(0) = \int_{-\infty}^\infty \frac{f(h(\varepsilon))^2 h'(\varepsilon)}{g(k(\phi(\varepsilon)))k'(\phi(\varepsilon))}dh = \frac{\sigma_\eta}{\sigma_\varepsilon}\int_{-\infty}^\infty \frac{f(h(\varepsilon))h'(\varepsilon)}{k'(\phi(\varepsilon))}dh.$$

Changing variables again, back from h to ε,

$$B'(0) = \frac{\sigma_\eta}{\sigma_\varepsilon}\int_0^\infty \frac{\tilde{f}(\varepsilon)h'(\varepsilon)}{k'(\phi(\varepsilon))}d\varepsilon.$$

Specializing to the lognormal case, suppose $h(\varepsilon) = \ln(\varepsilon)$ and $k(\eta) = \ln\eta$. Thus $h'(\varepsilon) = 1/\varepsilon$ and

$$B'(0) = \frac{\sigma_\eta}{\sigma_\varepsilon}\int_0^\infty \frac{\tilde{f}(\varepsilon)\phi(\varepsilon)}{\varepsilon}d\varepsilon = c_B'(x_B^*).$$

Suppose that $\mu_\varepsilon = \mu_\eta = 0$. Since $F(h(\varepsilon)) = G(k(\eta))$, $\ln(\varepsilon)^{1/\sigma_\varepsilon} = \ln[\phi(\varepsilon)]^{1/\sigma_\eta}$, so that $\phi(\varepsilon) = \varepsilon^{\sigma_\eta/\sigma_\varepsilon}$. The first-order condition for the boys is

$$\frac{\sigma_\eta}{\sigma_\varepsilon}\int_0^\infty \tilde{f}(\varepsilon)\varepsilon^{(\sigma_\eta/\sigma_\varepsilon)-1}d\varepsilon = c_B'(x_B^*).$$

When ε is lognormally distributed, the expectation of ε^α equals $\exp(\alpha\mu_\varepsilon + \frac{1}{2}\alpha^2\sigma_\varepsilon^2)$. Since $\mu_\varepsilon = 0$,

$$\frac{\sigma_\eta}{\sigma_\varepsilon}\exp\left[\frac{1}{2}\left(\frac{\sigma_\eta-\sigma_\varepsilon}{\sigma_\varepsilon}\right)^2\sigma_\varepsilon^2\right]=\frac{\sigma_\eta}{\sigma_\varepsilon}\exp\left[\frac{1}{2}(\sigma_\eta-\sigma_\varepsilon)^2\right]=c_B'(x_B^*).$$

Let us now consider large deviations. For $\Delta<0$, with probability $\frac{1}{2}\tilde{F}(\Delta)$, a downward deviation will result in being unmatched and is unattractive if $\bar{u}$ is low enough. For $\Delta>0$,

$$B'(\Delta)=\frac{\sigma_\eta}{\sigma_\varepsilon}\int_0^\infty \tilde{f}(\varepsilon)(\varepsilon+\Delta)^{(\sigma_\eta/\sigma_\varepsilon)-1}d\varepsilon.$$

Define $\alpha=(\sigma_\eta/\sigma_\varepsilon)-1$. When $\alpha\leq 0$, $B'(\Delta)$ is clearly concave in Δ. So let us consider the case in which $\alpha\in(0,1)$, so that the variance of the log of the shocks in the girls is less than twice that in the boys. A second-order Taylor expansion of $(\varepsilon+\Delta)^\alpha$ around $\Delta=0$ yields

$$(\varepsilon+\Delta)^\alpha=\varepsilon^\alpha+\alpha\varepsilon^{\alpha-1}\Delta+\alpha(\alpha-1)\varepsilon^{\alpha-2}\delta^2,$$

for some $\delta\in(0,\Delta]$. Thus,

$$\begin{aligned}B'(\Delta)&=\frac{\sigma_\eta}{\sigma_\varepsilon}[\mathbf{E}(\varepsilon^\alpha)+\alpha\mathbf{E}(\varepsilon^{\alpha-1})\Delta+\alpha(\alpha-1)\mathbf{E}(\varepsilon^{\alpha-2})\delta^2]\\&\leq\frac{\sigma_\eta}{\sigma_\varepsilon}[\mathbf{E}(\varepsilon^\alpha)+\alpha\mathbf{E}(\varepsilon^{\alpha-1})\Delta]\end{aligned}$$

and

$$c_B'(x_B^*+\Delta)=c_B'(x_B^*)+c_B''(x_B^*)\Delta+c_B'''(x_B^*)\delta^2,\quad \delta\in[0,\Delta].$$

Assume that $c_B'''(x_B^*)\geq 0$. Thus the problem is concave in Δ if

$$\frac{\sigma_\eta}{\sigma_\varepsilon}\alpha\mathbf{E}(\varepsilon^{\alpha-1})\Delta-c_B''(x_B^*)\Delta\leq 0.$$

This is satisfied if

$$c_B''(x_B^*)\geq\frac{\sigma_\eta}{\sigma_\varepsilon}\left(\frac{\sigma_\eta-\sigma_\varepsilon}{\sigma_\varepsilon}\right)\exp\left[\frac{1}{2}(\sigma_\eta-2\sigma_\varepsilon)^2\right].$$

In the boundary case in which $\sigma_\eta=2\sigma_\varepsilon$, the condition reduces to $c_B''(x_B^*)\geq 2$. Thus as long as the ratio of the variances is less than two and the second derivative of the cost function exceeds two at $x\geq x_B^*$, the problem is globally concave.

Proof of Proposition 2

Existence can be established by following the proof of theorem 1 except in the case of downward deviations by girls. A girl choosing $x_G^*+\Delta$ for some $\Delta<0$ faces a marginal benefit of

$$B'(\Delta)=\int_{\underline{\eta}}^{\bar{\eta}+\Delta}\frac{g(\eta)}{f(\phi^{-1}(\eta))}g(\eta-\Delta)d\eta.$$

We can find a distribution $F(\varepsilon)$ sufficiently dispersed in the sense of the dispersion order (see eq. [12] for a definition) so that $f(\cdot)$ is small enough to ensure that $B'(\Delta) > c_G'(x_G^* + \Delta)$ for $\Delta < 0$.

Turning to the efficiency of investments, we have $f(\tilde{\varepsilon})(\underline{\eta} + x_G^* - \bar{u}) > 0$, reflecting our assumption that the misery effect is strictly positive. Thus, combining the first-order conditions (14) and (15), we have

$$c_B'(x_B^*) \times c_G'(x_G^*) > \left[\frac{1}{r}\int_{\tilde{\varepsilon}(r)}^{\bar{\varepsilon}} \frac{f(\varepsilon)}{g(\phi(\varepsilon))} f(\varepsilon)\, d\varepsilon\right]\left[r\int_{\underline{\eta}}^{\bar{\eta}} \frac{g(\eta)}{f(\phi^{-1}(\eta))} g(\eta)\, d\eta\right].$$

Making a change of variables, from η to ε, results in

$$c_B'(x_B^*) \times c_G'(x_G^*) > \frac{1}{r}\left[\int_{\tilde{\varepsilon}(r)}^{\bar{\varepsilon}} \frac{f(\varepsilon)}{g(\phi(\varepsilon))} f(\varepsilon)\, d\varepsilon\right]\left[\int_{\tilde{\varepsilon}(r)}^{\bar{\varepsilon}} g(\phi(\varepsilon))\, d\varepsilon\right].$$

By the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned} &\frac{1}{r}\left[\int_{\tilde{\varepsilon}(r)}^{\bar{\varepsilon}} \frac{f(\varepsilon)}{g(\phi(\varepsilon))} f(\varepsilon)\, d\varepsilon\right]\left[\int_{\tilde{\varepsilon}(r)}^{\bar{\varepsilon}} g(\phi(\varepsilon))\, d\varepsilon\right] \\ &\geq \frac{1}{r}\left(\int_{\tilde{\varepsilon}(r)}^{\bar{\varepsilon}} \left\{\frac{f(\varepsilon)}{[g(\phi(\varepsilon))]^{1/2}}\right\}[g(\phi(\varepsilon))]^{1/2} d\varepsilon\right)^2 = \frac{r^2}{r}. \end{aligned}$$

Thus, $c_B'(x_B^*) \times c_G'(x_G^*) > r$ while efficiency requires equality. QED

Proof of Theorem 3

Let f and g be symmetric functions around their means, $\tilde{\varepsilon}$ and $\tilde{\eta}$. Symmetry implies that $f(\tilde{\varepsilon} - \Delta) = f(\tilde{\varepsilon} + \Delta)$ for any Δ. If f and g are both symmetric, then $\phi(\varepsilon)$ and $g(\phi(\varepsilon))$ are also symmetric around $\phi(\tilde{\varepsilon}) = \tilde{\eta}$ and $g(\phi(\tilde{\varepsilon}))$, respectively, and $f(\varepsilon)/g(\phi(\varepsilon))$ is also symmetric around $f(\tilde{\varepsilon})/g(\phi(\tilde{\varepsilon}))$. Using these facts,

$$\begin{aligned} \int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \varepsilon \frac{[f(\varepsilon)]^2}{g(\phi(\varepsilon)} d\varepsilon &= \int_{\underline{\varepsilon}-\tilde{\varepsilon}}^{0} (\tilde{\varepsilon} + \Delta)\frac{[f(\tilde{\varepsilon} + \Delta)]^2}{g(\phi(\tilde{\varepsilon} + \Delta))} d\Delta \\ &\quad + \int_{\underline{\varepsilon}-\tilde{\varepsilon}}^{0} (\tilde{\varepsilon} - \Delta)\frac{[f(\tilde{\varepsilon} - \Delta)]^2}{g(\phi(\tilde{\varepsilon} - \Delta))} d\Delta \\ &= 2\tilde{\varepsilon}\int_{\underline{\varepsilon}-\tilde{\varepsilon}}^{0} \frac{[f(\tilde{\varepsilon} + \Delta)]^2}{g(\phi(\tilde{\varepsilon} + \Delta))} d\Delta \\ &= 2\tilde{\varepsilon}\int_{\underline{\varepsilon}}^{\tilde{\varepsilon}} \frac{[f(\varepsilon)]^2}{g(\phi(\varepsilon))} d\varepsilon. \end{aligned}$$

Similarly,

$$\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \frac{[f(\varepsilon)]^2}{g(\phi(\varepsilon))}\, d\varepsilon = 2\int_{\underline{\varepsilon}}^{\tilde{\varepsilon}} \frac{[f(\varepsilon)]^2}{g(\phi(\varepsilon))}\, d\varepsilon.$$

A similar argument and a change of variables yield

$$\int_{\underline{\eta}}^{\bar{\eta}} \eta \frac{[g(\eta)]^2}{f(\phi^{-1}(\eta))} d\eta = \int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \phi(\varepsilon) g(\phi(\varepsilon)) d\varepsilon = 2\phi(\tilde{\varepsilon}) \int_{\underline{\varepsilon}}^{\bar{\varepsilon}} g(\phi(\varepsilon)) d\varepsilon.$$

We may therefore write the product of the equilibrium marginal costs as

$$\begin{gathered} \left\{ 2[\theta_G + (1-\theta_G)\gamma'(x_G^*)\tilde{\eta}] \int_{\underline{\varepsilon}}^{\varepsilon} g(\phi(\varepsilon)) d\varepsilon \right\} \\ \times \left\{ 2[\theta_B + (1-\theta_B)\gamma'(x_B^*)\tilde{\varepsilon}] \int_{\underline{\varepsilon}}^{\tilde{\varepsilon}} \frac{[f(\varepsilon)]^2}{g(\phi(\varepsilon))} d\varepsilon \right\}. \end{gathered} \tag{A14}$$

By the Cauchy-Schwarz inequality, the above is weakly greater than

$$4[\theta_G + (1-\theta_G)\gamma'(x_G^*)\tilde{\eta}][\theta_B + (1-\theta_B)\gamma'(x_B^*)\tilde{\varepsilon}] \left[\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} f(\varepsilon) d\varepsilon \right]^2,$$

which is equal to

$$[\theta_G + (1-\theta_G)\gamma'(x_G^*)\tilde{\eta}][\theta_B + (1-\theta_B)\gamma'(x_B^*)\tilde{\varepsilon}].$$

Thus the product of marginal costs is strictly greater than for Pareto efficiency if $f(\varepsilon)/\sqrt{g(\phi(\varepsilon))}$ and $\sqrt{g(\phi(\varepsilon))}$ are linearly independent. QED

Proof of Theorem 4

Under assumption 1, since costs are strictly convex and quality is concave in x, there is a unique profile of investments that satisfies the first-order conditions for maximizing utilitarian payoffs; assumption 1 also ensures that maximizing investments must be in the interior of the feasible set and must thus satisfy the first-order conditions. Since we have established that the first-order conditions for utilitarian efficiency are identical to the first-order conditions for an equilibrium that is gender-neutral, a gender-neutral profile can be an equilibrium only if it is utilitarian efficient.[25]

We now show that under the stated assumptions, deviations from the utilitarian profile are unprofitable, so that the utilitarian profile is an equilibrium. Given the convexity of the cost function and $q_{xx} \leq 0$, the utilitarian investments x_i^{**} globally maximize the utilitarian payoffs, implying

$$\int q(x,\varepsilon) f_i(\varepsilon) d\varepsilon - c_i(x) \leq \int q(x_i^{**},\varepsilon) f_i(\varepsilon) d\varepsilon - c_i(x_i^{**}) \quad \forall x. \tag{A15}$$

[25] Note that because of the strong assumption of symmetry between the sexes, we do not need assumption 3, which was required to assure uniqueness in the general asymmetric case.

Since each type chooses x_i^{**} in any gender-neutral equilibrium, the payoff to any individual of type i equals the right-hand side of (A15). Thus if the payoff to the individual from deviating to any $x \neq x_i^{**}$ is less than or equal to the left-hand side of (A15), no deviation is profitable. We show that this is the case if there are no quality gaps under the utilitarian profile and if $\bar{u}$ is sufficiently small. Let $C(\tilde{F})$ denote the support of $\tilde{F}$ under the utilitarian profile, which is connected under the no quality gap assumption. If a deviating individual's $q(x, \varepsilon) \in C(\tilde{F})$, then his or her match payoff equals $q(x, \varepsilon)$. If $q(x, \varepsilon) > \max C(\tilde{F})$, then the match payoff equals $\max C(\tilde{F})$. If $q(x, \varepsilon) < \min C(\tilde{F})$, then the match payoff equals $[\min C(\tilde{F}) + \bar{u}]/2$, which is less than $q(x, \varepsilon)$ if $\bar{u}$ is sufficiently small. Thus no deviation from x_i^{**} is profitable. QED

Proofs for the Finite Case

Our main result for the finite model, theorem 5, is the existence of a quasi-symmetric equilibrium and its convergence to the quasi-symmetric equilibrium of the continuum model. Our strategy of proof for this result is as follows. First, we consider the decision problem of a boy in a quasi-symmetric equilibrium. We fix a profile in which every girl chooses investment x_G and every other boy chooses investment x_B, and we consider the benefit function of a boy who chooses $x_B + \Delta$. This benefit function as given in (26) is written $B_n(\Delta, x_G)$. As we assume additive quality, B_n does not depend on x_B, but only on Δ and x_G, and is linear in x_G. From the individual boy's point of view, x_G is exogenously given, while Δ is a choice variable. We show in lemma 4 below that this benefit function converges to the benefit function in the continuum case, $B(\Delta, x_G)$, uniformly in (Δ, x_G), as $n \to \infty$. Lemma 4 also shows that the first derivative with respect to Δ evaluated at $\Delta = 0$, $B_n'(0, x_G)$, converges to $B'(0, x_G)$ for any value of x_G ($B'(0, x_G)$ is a constant function of x_G). Since $B_n'(0, x_G)$ is linear in x_G and since x_G is bounded, the convergence of $B_n'(0, x_G)$ is uniform in x_G. The first derivative, $B_n'(0, x_G)$, defines the "best-response" function for boys, $\hat{x}_{Bn}(x_G)$, and similarly, the first derivative for girls defines their "best-response" function, $\hat{x}_{Gn}(x_B)$. It is shown in lemma 5 that the convergence of the first derivatives to their continuum values implies that for n large enough, the best-response functions have positive slope less than one-quarter (the continuum best-response functions have slope zero), so that there exists a unique profile (x_{Bn}^*, x_{Gn}^*) that are mutual best responses. Furthermore, uniform convergence of the first derivatives implies that $(x_{Bn}^*, x_{Gn}^*) \to (x_B^*, x_G^*)$, the continuum equilibria, as $n \to \infty$.

It remains to verify that no individual boy has an incentive to deviate from x_{Bn}^*. Since $B_n(\Delta, x_G)$ converges uniformly to $B(\Delta, x_G)$, $(x_{Bn}^*, x_{Gn}^*) \to (x_B^*, x_G^*)$ as $n \to \infty$; and since large deviations are unprofitable in the continuum game, they are also unprofitable in the finite case when n is large enough. To show that small deviations are unprofitable, we establish two additional uniform convergence results, lemmas 6 and 7, for the second derivatives of $B_n(\Delta, x_G)$, for upward and downward deviations. Since $B_n(\Delta, x_G)$ is strictly concave for Δ in a neighborhood around zero, this establishes that no local deviations are profitable when n is large enough.

It might be worthwhile, before proceeding to the proof, to explain some of the issues involved in establishing uniform convergence. Lemma 1 shows that the matching function in the finite case, $\phi_n(\varepsilon + \Delta)$, converges pointwise to $\phi(\varepsilon + \Delta)$, the matching function in the continuum model (except at a single point, $\underline{\varepsilon}$). However, the finite agent matching function (see [A16]) is continuous on its domain, while the continuum matching function is discontinuous to the left at $\underline{\varepsilon}$. Consequently, by Weirstrass's theorem, convergence cannot be uniform.[26] Fortunately, we do not require uniform convergence of the matching function, but of the payoff function $B_n(\cdot)$, which is the integral of the matching functions (see lemmas 3 and 4 below).

Define $\phi(\varepsilon + \Delta)$, the shock value of the girl who is matched with a boy who invests $x_B + \Delta$ and receives shock ε, by

$$\phi(\varepsilon + \Delta) = \begin{cases} \frac{1}{2}(\underline{\eta} + \bar{u} - x_G) & \text{if } \varepsilon + \Delta < \underline{\varepsilon} \\ G^{-1}(F(\varepsilon)) & \text{if } \varepsilon + \Delta \in [\underline{\varepsilon}, \bar{\varepsilon}] \\ \bar{\eta} & \text{if } \varepsilon + \Delta > \bar{\varepsilon}. \end{cases}$$

Similarly, let $\tilde{\phi}_n(\varepsilon + \Delta)$ denote the shock value of the girl who is matched with a boy who invests $x_B + \Delta$ and receives shock ε. Let $\phi_n(\varepsilon + \Delta)$ denote the expectation of $\tilde{\phi}_n(\varepsilon + \Delta)$. As we have already discussed in the text, this equals the expected shock value of a girl who is matched with a boy of shock value $\varepsilon + \Delta$, with the caveat that if $\varepsilon + \Delta > \bar{\varepsilon}$, he marries the highest-quality girl, and if $\varepsilon + \Delta < \underline{\varepsilon}$, he marries the lowest-quality girl with probability $n/(2n + 1)$ and is unmarried with probability $(n + 1)/(2n + 1)$, in which case the shock value is defined to be $\bar{u} - x_G$. Also, $\phi_n(\varepsilon + \Delta)$ can be written as

$$\phi_n(\varepsilon + \Delta) = \frac{n}{2n + 1}\phi_{n,n+1}(\varepsilon + \Delta) + \frac{n + 1}{2n + 1}\phi_{n+1,n}(\varepsilon + \Delta), \tag{A16}$$

where $\phi_{n,n+1}(\varepsilon + \Delta)$ (respectively, $\phi_{n+1,n}(\varepsilon + \Delta)$) equals the expected shock value of the girl that the boy with shock $\varepsilon + \Delta$ is matched with, conditional on there being n boys and $n + 1$ girls (respectively, n girls and $n + 1$ boys). Specifically,

$$\phi_{n,n+1}(\varepsilon + \Delta) = \sum_{i=1}^{n} F_i^n(\varepsilon + \Delta)E\eta_{(i+1,n+1)}, \tag{A17}$$

where, using the notation of Hoppe et al. (2009), $F_i^n(\varepsilon)$ denotes the probability that a boy with shock ε is ranked i when there are n boys, and $E\eta_{(j,n+1)}$ is the expectation of the jth order statistic for shocks for girls when there are $n + 1$ girls. Similarly,

$$\phi_{n+1,n}(\varepsilon + \Delta) = \sum_{i=2}^{n+1} F_i^{n+1}(\varepsilon + \Delta)E\eta_{(i-1,n)} + F_1^{n+1}(\varepsilon + \Delta)(\bar{u} - x_G), \tag{A18}$$

where $F_i^{n+1}(\varepsilon + \Delta)$ and $E\eta_{(i-1,n)}$ are defined analogously. The final term represents the probability of coming last and remaining unmatched.

For our convergence results, we shall restrict Δ to lie in the compact interval $[-d, d]$ that contains $[\underline{\Delta}, \bar{\Delta}]$. We consider an arbitrary but larger interval than

[26] This does not preclude uniform convergence of the matching function on a restricted domain where the continuum matching function is continuous, as we show in lemma 2.

$[\underline{\Delta}, \bar{\Delta}]$ in order to show that deviations outside $[\underline{\Delta}, \bar{\Delta}]$ are unprofitable in the finite model just as in the continuum case. The following lemma establishes the pointwise convergence of the matching function ϕ_n that is the average of $\phi_{n,n+1}$ and $\phi_{n+1,n}$. Since $\phi_{n,n+1}$ is well behaved for upward deviations (even though $\phi_{n+1,n}$ is not), it also establishes its pointwise convergence in this case.

LEMMA 1. Let $\Delta \in [-d, d]$, where $d > 0$ is arbitrary. For any $\varepsilon + \Delta \in [\underline{\varepsilon} - d, \bar{\varepsilon} + d]$ except $\varepsilon + \Delta = \underline{\varepsilon}$, $\phi_n(\varepsilon + \Delta)$ converges pointwise to $\phi(\varepsilon + \Delta)$ as $n \to \infty$. Moreover, $\phi_{n,n+1}(\varepsilon + \Delta)$ converges pointwise to $\phi(\varepsilon + \Delta)$ at any $\varepsilon + \Delta \in [\underline{\varepsilon}, \bar{\varepsilon} + d]$.

Proof. Let $F_n(t)$ (respectively, $G_n(t)$) denote the fraction of boys (respectively, girls) with shock value below t in a sample of n boys (girls), where $t \in (\underline{\varepsilon}, \bar{\varepsilon})$. By the Glivenko-Cantelli theorem, $F_n(t) \to F(t)$ almost surely as $n \to \infty$. Similarly, $G_n(t) \to G(t)$ almost surely as $n \to \infty$. Thus for any a, b such that $\underline{\varepsilon} < a < \varepsilon + \Delta < b < \bar{\varepsilon}$, the probability that $\tilde{\phi}(\varepsilon + \Delta) \in (\phi(a), \phi(b))$ tends to one as $n \to \infty$. Hence $\phi_n(\varepsilon + \Delta) \in (\phi(a), \phi(b))$ for all n sufficiently large. Consequently, since a and b were arbitrary, $\phi_n(\varepsilon + \Delta) \to \phi(\varepsilon + \Delta)$ if $\underline{\varepsilon} < \varepsilon + \Delta < \bar{\varepsilon}$. Also, by definition, if $\varepsilon + \Delta \geq \bar{\varepsilon}$,

$$\phi_n(\varepsilon + \Delta) = \frac{n}{2n+1} E\eta_{(n+1,n+1)} + \frac{n+1}{2n+1} E\eta_{(n,n)},$$

which converges to $\bar{\eta} = \phi(\varepsilon + \Delta)$. Finally, when $\varepsilon + \Delta < \underline{\varepsilon}$,

$$\phi_n(\varepsilon + \Delta) = \frac{n}{2n+1} E\eta_{(2,n+1)} + \frac{n+1}{2n+1} (\bar{u} - x_G).$$

Since $E\eta_{(2,n+1)} \to \underline{\eta}$, as $n \to \infty$, $\phi_n(\varepsilon + \Delta)$ converges to $\frac{1}{2}(\underline{\eta} + \bar{u} - x_G) = \phi(\varepsilon + \Delta)$ as $n \to \infty$. Hence $\phi_n(\varepsilon + \Delta)$ converges pointwise to $\phi(\varepsilon + \Delta)$ for all values of $\varepsilon + \Delta$ except $\varepsilon + \Delta = \underline{\varepsilon}$. The argument for $\phi_{n,n+1}(\varepsilon + \Delta)$ is identical for $\varepsilon + \Delta > \underline{\varepsilon}$. At $\varepsilon + \Delta = \underline{\varepsilon}$, $\phi_{n,n+1}(\varepsilon + \Delta) = E\eta_{(2,n+1)}$, which converges to $\underline{\eta}$ as $n \to \infty$. Since $\phi(\underline{\varepsilon}) = \underline{\eta}$, $\phi_{n,n+1}(\varepsilon + \Delta) \to \phi(\varepsilon + \Delta)$ for any $\varepsilon + \Delta \in [\underline{\varepsilon}, \bar{\varepsilon} + a]$. QED

Note that $\phi_n(\underline{\varepsilon})$ does not converge to $\phi(\underline{\varepsilon})$. Thus one cannot expect uniform convergence of the matching function in general, but the following lemma shows uniform convergence on a restricted domain.

LEMMA 2. Let $0 < b < d$, and let $\Delta \in [b, d]$. The function $\phi_n(\varepsilon + \Delta)$ converges to $\phi(\varepsilon + \Delta)$ as $n \to \infty$, uniformly on $\varepsilon + \Delta \in [\underline{\varepsilon} + b, \bar{\varepsilon} + d]$. Further, $\phi_{n,n+1}(\varepsilon + \Delta)$ converges to $\phi(\varepsilon + \Delta)$ uniformly in Δ on $[\underline{\varepsilon}, \bar{\varepsilon} + d]$.

Proof. Lemma 1 establishes that $\phi_n(\varepsilon + \Delta)$ converges to $\phi(\varepsilon + \Delta)$ at each value except $\varepsilon + \Delta = \underline{\varepsilon}$. Since $\phi_n(\varepsilon + \Delta)$ is a strictly increasing and continuous function and it converges to a continuous function on the compact domain $[\underline{\varepsilon} + b, \bar{\varepsilon} + d]$, convergence is uniform (as an immediate consequence of Polya's theorem). Similarly, lemma 1 shows that $\phi_{n,n+1}(\varepsilon + \Delta)$ converges to $\phi(\varepsilon + \Delta)$ at any $\varepsilon + \Delta \in [\underline{\varepsilon}, \bar{\varepsilon} + d]$, and again Polya's theorem implies uniform convergence. QED

We move on to the convergence of the benefit function B_n, as introduced in (26), and its derivatives. The following lemma ensures that establishing pointwise convergence of the matching functions ensures uniform convergence of the benefit function.

Let $\xi : \mathbb{R} \to \mathbb{R}$ equal zero except on a compact interval. Without loss of generality, let this interval be $[0, 1]$, so that $\xi(x) = 0$ for all $x \notin [0, 1]$. Let $h : [-a, 1 + a] \to \mathbb{R}$, and for $n \in \mathbb{N}$, let $h_n : [-a, 1 + a] \to \mathbb{R}$. Assume that each of these functions ξ, h, and h_n are measurable and bounded, and let $\bar{\xi} > 0$ be an upper bound for $|\xi|$. For $\Delta \in [-a, a]$, define $W_n(\Delta)$ and $W(\Delta)$ by

$$W_n(\Delta) = \int_0^1 h_n(x+\Delta)\xi(x)dx,$$

$$W(\Delta) = \int_0^1 h(x+\Delta)\xi(x)dx.$$

LEMMA 3. Assume that for almost any $z \in [-a, 1+a]$, $h_n(z) \to h(z)$ as $n \to \infty$; that is, h_n converges pointwise to h almost everywhere on $[-a, 1+a]$. Assume also that $h_n(z)$ is uniformly bounded; that is, $|h_n(z)| < H$ for all n and for all $z \in [-a, 1+a]$. Then, W_n converges to $W(\Delta)$ uniformly in Δ.

Proof. Using the change of variables $s = x + \Delta$, we have

$$W_n(\Delta) = \int_{\Delta}^{1+\Delta} h_n(s)\xi(s-\Delta)ds;$$
$$W(\Delta) = \int_{\Delta}^{1+\Delta} h(s)\xi(s-\Delta)ds.$$

Hence,

$$\begin{aligned} |W_n(\Delta) - W(\Delta)| &= \left| \int_{\Delta}^{1+\Delta} [h_n(s) - h(s)]\xi(s-\Delta)ds \right| \\ &\le \int_{\Delta}^{1+\Delta} |h_n(s) - h(s)|\,|\xi(s-\Delta)|ds \qquad \text{(A19)} \\ &\le \bar{\xi}\int_{-a}^{1+a} |h_n(s) - h(s)|ds. \end{aligned}$$

Since the last expression, which does not depend on Δ, converges to zero by the dominated convergence theorem, the conclusion follows. QED

With additive quality, the benefit function $B(\Delta, x_G)$ as introduced in (A2) and its finite equivalent $B_n(\Delta, x_G)$ are given by

$$\begin{aligned} B(\Delta, x_G) &= \int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \phi(\varepsilon+\Delta)f(\varepsilon)\ d\varepsilon + x_G, \\ B_n(\Delta, x_G) &= \int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \phi_n(\varepsilon+\Delta)f(\varepsilon)\ d\varepsilon + x_G. \end{aligned} \qquad \text{(A20)}$$

We now write B as a function of (Δ, x_G) to emphasize that convergence of the finite equivalent B_n is joint in both variables (B is not a function of x_B, however). In the proof of theorem 1 we established that since $f(\underline{\varepsilon}) = 0$, the left-hand and right-hand derivatives with respect to Δ of $B(\Delta, x_G)$, evaluated at $\Delta = 0$, are equal. Thus, the derivative exists and equals

$$B'(0, x_G) = \int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \phi'(\varepsilon) f(\varepsilon) d\varepsilon.$$

Note that $B'(0, x_G)$ does not depend on x_G. The derivative of $B_n(\cdot)$ with respect to Δ is

$$B_n'(0, x_G) = \int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \phi_n'(\varepsilon) f(\varepsilon) d\varepsilon. \tag{A21}$$

Since quality is additive, $B(\Delta, x_G)$, $B_n(\Delta, x_G)$, and their derivatives are all linear in x_G.

LEMMA 4. $B_n(\Delta, x_G) \to B(\Delta, x_G)$ uniformly in (Δ, x_G) as $n \to \infty$. Further, $B_n'(0, x_G) \to B'(0, x_G)$ uniformly in x_G as $n \to \infty$.

Proof. In the expression for $B_n(\Delta, x_G)$ in (A20), the integrand is uniformly bounded: below by $\bar{u} - x_G$ and above by $\bar{\eta}$. Thus lemma 3 and lemma 1 imply that $B_n(\Delta, x_G)$ converges to $B(\Delta, x_G)$ uniformly in Δ. The result that it converges uniformly in (Δ, x_G) follows since B_n is linear in x_G and convergence is on the compact set $[-a, a] \times [0, \tilde{x}_G]$.

Use integration by parts and the fact that $f(\underline{\varepsilon}) = 0$ to rewrite the expression for $B'(0, x_G)$ as

$$B'(0, x_G) = -\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \phi(\varepsilon) f'(\varepsilon) d\varepsilon + \bar{\eta} f(\bar{\varepsilon}). \tag{A22}$$

Similarly, using integration by parts, we find that

$$B_n'(0, x_G) = -\int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \phi_n(\varepsilon) f'(\varepsilon) d\varepsilon + \phi_n(\bar{\varepsilon}) f(\bar{\varepsilon}). \tag{A23}$$

Since $f'(\varepsilon)$ is bounded, lemma 3 and lemma 1 imply that the first term in (A23) converges to the first term in (A22). Lemma 1 implies that the second term in (A23) converges to the second term in (A22). Uniform convergence in x_G again follows from the linearity in x_G and its boundedness. QED

Let $\tilde{B}_n(\Delta, x_B)$ denote the benefit function for girls, and let $B_n'(0, x_B)$ denote its derivative at $\Delta = 0$.

LEMMA 5. For n sufficiently large, there exists a unique profile (x_{Bn}^*, x_{Gn}^*), where the first-order conditions are satisfied, for the boys, $c_B'(x_{Bn}^*) = B_n'(0, x_{Gn}^*)$, and for the girls, $c_G'(x_{Gn}^*) = B_n'(0, x_{Bn}^*)$. Furthermore, $(x_{Bn}^*, x_{Gn}^*) \to (x_B^*, x_G^*)$, the unique quasi-symmetric equilibrium of continuum economy.

Proof. Define $\hat{x}_{Bn}(x_G)$ and $\hat{x}_{Gn}(x_B)$ by

$$c_B'(\hat{x}_{Bn}) = B_n'(0, x_G),$$

$$c_G'(\hat{x}_{Gn}) = B_n'(0, x_B).$$

Since $c_B'(0) < 0 < B_n'(0, x_G)$ and $c_B'(x_B) \to \infty$ as $x_B \to \tilde{x}_B$, and since $c_B'(\cdot)$ is strictly increasing, $\hat{x}_{Bn}(x_{Gn})$ is uniquely defined. Its derivative is

$$\frac{d\hat{x}_{Bn}}{dx_G} = \frac{\partial B_n'(0, x_G)}{\partial x_G} \frac{1}{c_B''(\hat{x}_{Bn})}. \tag{A24}$$

Note that $B'(0, x_G)$ does not depend on x_G, while $B_n'(0, x_G)$ is linear in x_G and converges uniformly to $B'(0, x_G)$. Thus the derivative $\partial B_n'(0, x_G)/\partial x_G \to 0$ as $n \to \infty$. Since $c_B''(\hat{x}_{Bn}) \geq \gamma > 0$ (cf. assumption 1), $d\hat{x}_{Bn}/dx_G \to 0$ as $n \to \infty$. Since the argument is identical for the girls, $d\hat{x}_{Gn}/dx_B \to 0$ as $n \to \infty$. Thus there exists $N : n > N \Rightarrow d\hat{x}_{Bn}/dx_G < \frac{1}{2}$ for any $x_G \in [0, \tilde{x}_G]$ and $d\hat{x}_{Gn}/dx_B < \frac{1}{2}$ for any $x_B \in [0, \tilde{x}_B]$.

Define $\zeta_n : [0, \tilde{x}_B] \to [0, \tilde{x}_B]$ by $\zeta_n(x) = \hat{x}_{Bn}(\hat{x}_{Gn}(x))$. Since ζ_n is a composition of increasing differentiable functions, it is increasing and differentiable, with derivative equal to the product of the derivatives of $\hat{x}_{Bn}(\cdot)$ and $\hat{x}_{Gn}(\cdot)$. Thus if $n > N$, where N is as defined in the previous paragraph, ζ_n has a slope that is bounded above by ¼, and so any fixed point is unique. Since $\hat{x}_{Bn}(x_G) \geq \bar{x}_B$, $\zeta_n(0) > 0$; and since $\zeta_n(\tilde{x}_B) < \tilde{x}_B$, the intermediate value theorem ensures existence of a fixed point of ζ_n, x^*_{Bn}. Let $x^*_{Gn} = \hat{x}_{Gn}(x^*_{Bn})$. Thus if $n > N$, there exists a unique profile (x^*_{Bn}, x^*_{Gn}), where the first-order conditions are satisfied.

We now show that sequence $(x^*_{Bn}, x^*_{Gn})_{n=N+1}^{\infty}$ converges to (x^*_B, x^*_G), the unique equilibrium in the continuum model. Recall that x^*_B is defined by $c_B'(x^*_B) = B'(0, x_G)$, where $B'(0, x_G)$ is a constant, so that $x^*_B = (c_B')^{-1}(B'(0, x_G))$, where $(c_B')^{-1}$ denotes the inverse of the marginal cost function. On the other hand, $x^*_{Bn} = (c_B')^{-1}(B_n'(0, x^*_{Gn}))$. Let x_{Gn} be an arbitrary sequence that takes values in $[0, \tilde{x}_G)$, and consider the induced sequence $x_{Bn} \coloneqq (c_B')^{-1}(B_n'(0, x_{Gn}))$. Since $B_n'(0, x_{Gn})$ converges uniformly to the constant $B'(0, x_G)$ uniformly in x_{Gn} and since $(c_B')^{-1}$ is a continuous function (since $c_B'(x)$ is continuous and strictly increasing), $\lim_{n\to\infty} x_{Bn} = (c_B')^{-1}(\lim_{n\to\infty} B_n'(0, x_{Gn})) = x^*_B$. Thus $x_{Bn} \to x^*_B$ as $n \to \infty$, and since the sequence x_{Gn} was arbitrary, this proves that $x^*_{Bn} = (c_B')^{-1}(B_n'(0, x^*_{Gn})) \to x^*_B$ as $n \to \infty$. Similarly, $x^*_{Gn} \to x^*_G$ as $n \to \infty$. QED

We now show that deviations from (x^*_{Bn}, x^*_{Gn}) are unprofitable if n is large enough. Our strategy is to show that if the global optimality conditions are satisfied in the continuum case, then they are also satisfied in the large finite case. For upward deviations, we show that the second derivative converges uniformly to the continuum derivative plus a negative term. For downward deviations in some interval, we do this by showing the uniform convergence of the second derivative to that in the continuum model.

In the continuum model, the second derivative on $(0, \bar{\Delta})$ given in (A7) becomes with additive quality

$$B''(\Delta, x_G) = \int_{\underline{\varepsilon}}^{\bar{\varepsilon}-\Delta} \phi''(\varepsilon + \Delta) f(\varepsilon) d\varepsilon - \phi'(\bar{\varepsilon}) f(\bar{\varepsilon} - \Delta). \tag{A25}$$

For $[-d, 0)$, the expression (A9) becomes,

$$\begin{aligned} B''(\Delta, x_G) = &\int_{\underline{\varepsilon}-\Delta}^{\bar{\varepsilon}} \phi''(\varepsilon + \Delta) f(\varepsilon) d\varepsilon + \phi'(\underline{\varepsilon}) f(\underline{\varepsilon} - \Delta) \\ &- f'(\underline{\varepsilon} - \Delta) \frac{x_G + \underline{\eta} - \bar{u}}{2}. \end{aligned} \tag{A26}$$

The second derivatives in the finite case are, for $\Delta\in(0,\bar{\Delta})$,

$$B_n''(\Delta, x_G) = \int_{\underline{\varepsilon}}^{\bar{\varepsilon}-\Delta} \phi_n''(\varepsilon+\Delta)f(\varepsilon)d\varepsilon - \phi_n'(\bar{\varepsilon})f(\bar{\varepsilon}-\Delta), \tag{A27}$$

and for $\Delta \in [-d, 0)$,

$$B_n''(\Delta, x_G) = \int_{\underline{\varepsilon}-\Delta}^{\bar{\varepsilon}} \phi_n''(\varepsilon+\Delta)f(\varepsilon)d\varepsilon + \phi_n'(\underline{\varepsilon})f(\underline{\varepsilon}-\Delta). \tag{A28}$$

At this point, for the convergence of $B_n''(\Delta, x_G)$ on $[-d, 0)$, we invoke an additional assumption, $f(\bar{\varepsilon}) = 0$. We will relax this assumption later.

Lemma 6. For any $k > 0$, there exists $N : n > N \Rightarrow B_n''(\Delta, x_G) < B''(\Delta, x_G) + k$ for any $\Delta \in (0, \bar{\Delta})$. The function $B_n''(\Delta, x_G) \to B''(\Delta, x_G)$ uniformly in (Δ, x_G) for $\Delta \in [-d, 0)$ as $n \to \infty$ if $f(\bar{\varepsilon}) = 0$.

Proof. For $\Delta \in (0, \bar{\Delta})$ we show that $B_n''(\Delta, x_G)$ can be written as the sum of two terms, where the first converges to $B''(\Delta, x_G)$ uniformly in (Δ, x_G) and the second is negative. By iterated integration by parts, we obtain

$$B''(\Delta, x_G) = \int_{\underline{\varepsilon}}^{\bar{\varepsilon}-\Delta} \phi(\varepsilon+\Delta)f''(\varepsilon)d\varepsilon - \phi(\bar{\varepsilon})f'(\bar{\varepsilon}-\Delta) + \phi(\underline{\varepsilon}+\Delta)f'(\underline{\varepsilon}), \tag{A29}$$

$$B_n''(\Delta, x_G) = \int_{\underline{\varepsilon}}^{\bar{\varepsilon}-\Delta} \phi_n(\varepsilon+\Delta)f''(\varepsilon)d\varepsilon - \phi_n(\bar{\varepsilon})f'(\bar{\varepsilon}-\Delta) + \phi_n(\underline{\varepsilon}+\Delta)f'(\underline{\varepsilon}). \tag{A30}$$

Since $f''(\varepsilon)$ is continuous on the support of f, it is bounded. Thus the first term of (A30) converges to the first term in (A29). Turning to the second terms in the two expressions, convergence follows since $\phi_n(\bar{\varepsilon})$ converges to $\phi(\bar{\varepsilon})$. These terms are multiplied by $f'(\bar{\varepsilon}-\Delta)$, but since f' is continuous and therefore bounded, the convergence is also uniform in Δ. Uniform convergence in x_G follows from the linearity of B_n'' in x_G.

This leaves the final term in (A30), $\phi_n(\underline{\varepsilon}+\Delta)f'(\underline{\varepsilon})$. Let $\tilde{\phi}_{n,n+1}(\underline{\varepsilon}+\Delta)$ be the random variable that denotes the shock value of the partner when there are $n+1$ girls and n boys. Similarly, let $\tilde{\phi}_{n+1,n}(\underline{\varepsilon}+\Delta)$ be the random variable that denotes the shock value of the partner when there are $n+1$ girls and n boys. Note that $\tilde{\phi}_{n,n+1}(\underline{\varepsilon}+\Delta)$ first-order stochastic dominates $\tilde{\phi}_{n+1,n}(\underline{\varepsilon}+\Delta)$,[27] and so $\phi_{n,n+1}(\underline{\varepsilon}+\Delta) > \phi_{n+1,n}(\underline{\varepsilon}+\Delta)$. Since $\phi_n(\underline{\varepsilon}+\Delta)$ is a convex combination of $\phi_{n,n+1}(\underline{\varepsilon}+\Delta)$ and $\phi_{n+1,n}(\underline{\varepsilon}+\Delta)$, it follows that $\phi_n(\underline{\varepsilon}+\Delta) < \phi_{n,n+1}(\underline{\varepsilon}+\Delta)$. Let $\tilde{B}_n''(\Delta, x_G)$ equal the

[27] Given $n-1$ realizations of draws from the distribution $F(\cdot)$, the absolute rank of $\varepsilon+\Delta$ in the set of n boys must be weakly lower when we have an additional draw from $F(\cdot)$. Similarly, the shock value of a girl of any absolute rank j in a set of $n+1$ girls must be weakly lower when we remove one of the girls. Thus $\tilde{\phi}_{n,n+1}(\underline{\varepsilon}+\Delta)$ first-order stochastic dominates $\tilde{\phi}_{n+1,n}(\underline{\varepsilon}+\Delta)$.

right-hand side of (A30), modified by replacing $\phi_n(\underline{\varepsilon}+\Delta)$ with $\phi_{n,n+1}(\underline{\varepsilon}+\Delta)$. By lemma 2, $\phi_{n,n+1}(\underline{\varepsilon}+\Delta)$ converges uniformly to $\phi(\underline{\varepsilon}+\Delta)$, and so $\tilde{B}''_n(\Delta, x_G)$ converges to $B''(\Delta, x_G)$ uniformly in Δ. Since $B''_n(\Delta, x_G) < \tilde{B}''_n(\Delta, x_G)$ and since for any $k>0$, for all $\Delta \in (0, \bar{\Delta})$, there exists $N: n > N \Rightarrow |\tilde{B}''_n(\Delta) - B''(\Delta)| < k$, implying that $B''_n(\Delta) < B''(\Delta) + k$.

Turning to $B''(\Delta, x_G)$ and $B''_n(\Delta, x_G)$ on $[-d, 0)$, iterated integration by parts yields

$$B''(\Delta, x_G) = \int_{\underline{\varepsilon}-\Delta}^{\bar{\varepsilon}} \phi(\varepsilon+\Delta) f''(\varepsilon) d\varepsilon + f'(\underline{\varepsilon}-\Delta)\left(\frac{x_G+\underline{\eta}+\bar{u}}{2}\right) - \phi(\bar{\varepsilon}+\Delta) f'(\bar{\varepsilon}) + \phi'(\bar{\varepsilon}+\Delta) f(\bar{\varepsilon}), \tag{A31}$$

$$B''_n(\Delta, x_G) = \int_{\underline{\varepsilon}-\Delta}^{\bar{\varepsilon}} \phi_n(\varepsilon+\Delta) f''(\varepsilon) d\varepsilon + \phi_n(\underline{\varepsilon}) f'(\underline{\varepsilon}-\Delta) - \phi_n(\bar{\varepsilon}+\Delta) f'(\bar{\varepsilon}) + \phi'_n(\bar{\varepsilon}+\Delta) f(\bar{\varepsilon}). \tag{A32}$$

Uniform convergence of the first two terms in (A32) to the first two terms in (A31) is by the same argument as for B'' when $\Delta > 0$. Uniform convergence of the third term in (A32) to the third term in (A31) follows from lemma 2. Given the assumption that $f(\bar{\varepsilon}) = 0$, the fourth terms in (A32) and (A31) both equal zero. QED

At this point, we could move directly to the proof of the main theorem if we assume that f and g equal zero at the upper bound of their supports. This assumption is not necessary, but relaxing it requires showing that $\phi'_n(\bar{\varepsilon}+\Delta)$ converges to $\phi'(\bar{\varepsilon}+\Delta)$ uniformly for Δ belonging to some interval $[-b, 0]$, as we now show.

Lemma 7. There exists $b>0$ such that, for $\Delta \in [-b, 0]$, $\phi'_n(\bar{\varepsilon}+\Delta)$ converges uniformly in Δ to $\phi'(\bar{\varepsilon}+\Delta)$ as $n \to \infty$. Hence, $B''_n(\Delta, x_G) \to B''(\Delta, x_G)$ uniformly in (Δ, x_G) for $\Delta \in [-b, 0)$ as $n \to \infty$.

Proof. Let $b \in (0, \bar{\varepsilon}-\underline{\varepsilon})$, so that $\bar{\varepsilon}+\Delta \in [\underline{\varepsilon}, \bar{\varepsilon}]$. Since

$$\phi'_n(\bar{\varepsilon}+\Delta) = \frac{n+1}{2n+1}\phi'_{n+1,n}(\bar{\varepsilon}+\Delta) + \frac{n}{2n+1}\phi'_{n,n+1}(\bar{\varepsilon}+\Delta),$$

it converges to $\phi'(\bar{\varepsilon}+\Delta)$ if both $\phi'_{n+1,n}(\bar{\varepsilon}+\Delta)$ and $\phi'_{n,n+1}(\bar{\varepsilon}+\Delta)$ converge. We demonstrate the convergence of $\phi'_{n+1,n}(\bar{\varepsilon}+\Delta)$ since the argument for $\phi'_{n,n+1}(\bar{\varepsilon}+\Delta)$ is almost identical. By differentiating (A18) we obtain

$$\phi'_{n+1,n}(\bar{\varepsilon}+\Delta) = f(\bar{\varepsilon}+\Delta)\sum_{i=2}^{n} F_i^n(\bar{\varepsilon}+\Delta) n[E\eta_{(i,n)} - E\eta_{(i-1,n)}] + f(\bar{\varepsilon}+\Delta) n[1-F(\bar{\varepsilon}+\Delta)]^{n-1}[E\eta_{(1,n)} + x_G - \bar{u}] > 0. \tag{A33}$$

Consider first the final term in the above expression. For any n, $n[1-F(\bar{\varepsilon}+\Delta)]^n \leq n[1-F(\bar{\varepsilon}-d)]^n$, and $n[1-F(\bar{\varepsilon}+\Delta)]^n \to 0$ as $n \to \infty$ for every $\Delta \in [-b, 0]$. Since $\Delta = b$ provides an upper bound, the convergence is uniform in Δ.

Turning to the summation terms in (A33), $f(\bar{\varepsilon}+\Delta)$ enters both $\phi'_{n+1,n}(\bar{\varepsilon}+\Delta)$ and $\phi'(\bar{\varepsilon}+\Delta)$, so it suffices to show that

$$\sum_{i=2}^{n} F_i^n(\bar{\varepsilon}+\Delta)n[E\eta_{(i,n)} - E\eta_{(i-1,n)}] \to \frac{1}{g(G^{-1}(F(\bar{\varepsilon}+\Delta)))}$$

as $n \to \infty$ uniformly in Δ.

Let the integer $i(n) = (n+1)p$ for p taking values in $[0, 1]$. We now show that the expression $n[E\eta_{(i(n),n)} - E\eta_{(i(n)-1,n)}]$ converges to $1/g(G^{-1}(p))$ as $n \to \infty$ uniformly in p for p belonging to an interval $[\tilde{p}, 1]$, where $g(G^{-1}(p))$ is bounded away from zero. Assume that $g(G^{-1}(1)) = g(\bar{\eta}) > 0$. Since g is continuous, there exists an interval $[\hat{p}, 1]$ such that $g(G^{-1}(p)) > \bar{g} > 0$ for all $p \in [\hat{p}, 1]$, and let $\tilde{p} \in (\hat{p}, 1)$. From Arnold, Balakrishnan, and Nagaraja (1992, 128) we have

$$E\eta_{(i,n)} = G^{-1}(p) + \frac{p(1-p)}{2(n+2)}\frac{d^2G^{-1}(p)}{du^2} + O\left(\frac{1}{n^2}\right),$$

where $dG^{-1}(p)/du$ is the derivative of $G^{-1}(u)$ evaluated at p. Furthermore, the terms that are $O(1/n^2)$ are so uniformly in p for $p \in [0, 1]$. Now,

$$\begin{aligned} n[E\eta_{(i,n)} - E\eta_{(i-1,n)}] = {} & n\left[G^{-1}(p) - G^{-1}\left(p - \frac{1}{n+1}\right)\right] \\ & + n\frac{d^2G^{-1}\left(p-\frac{1}{n+1}\right)}{du^2}\left[\frac{\frac{1-2p}{n+1} + \left(\frac{1}{n+1}\right)^2}{2(n+2)}\right] \\ & + n\frac{p(1-p)}{2(n+2)}\left[\frac{d^2G^{-1}(p)}{du^2} - \frac{d^2G^{-1}\left(p-\frac{1}{n+1}\right)}{du^2}\right] + nO\left(\frac{1}{n^3}\right). \end{aligned} \tag{A34}$$

We first show that the first term on the right-hand side converges uniformly to $1/g(G^{-1}(p))$ as $n \to \infty$ uniformly in p for $p \in [\tilde{p}, 1]$. By the mean value theorem, there exists $a \in (p - 1/(n+1), p)$ such that

$$[1/g(G^{-1}(a))]\frac{1}{n+1} = G^{-1}(p) - G^{-1}(p - 1/(n+1)).$$

Since $1/g(G^{-1}(\cdot))$ is continuous and since $1/(n+1) \to 0$ as $n \to \infty$,

$$[1/g(G^{-1}(a))]\frac{n}{n+1} \to 1/g(G^{-1}(p))$$

as $n \to \infty$. Convergence is uniform in p for $p \in [\tilde{p}, 1]$ since $1/g(G^{-1}(\cdot))$ is uniformly continuous on the compact interval $[\hat{p}, 1]$.

Let us now turn to other terms in (A34) to show that they go to zero as $n \to \infty$ uniformly in p, for $p \in (p, 1)$. This is true for

$$n\left[\frac{\dfrac{1-2p}{n+1}-\left(\dfrac{1}{n+1}\right)^2}{2(n+2)}\right]$$

since it is of order $1/n$, and $p = 1$ provides an upper bound for its absolute value for large n. Turning to the third term, our lower bound on g implies that

$$\frac{d^2G^{-1}(p)}{du^2}-\frac{d^2G^{-1}\left(p-\dfrac{1}{n+1}\right)}{du^2}$$

is $O(1/n)$ uniformly in p, and thus this term also converges to zero as $n \to \infty$ uniformly in p, for $p \in (\tilde{p}, 1)$. The final term is of order $1/n^3$ since the difference between the two $O(1/n^2)$ terms is $O(1/n^3)$.

Consider any $p \in (\tilde{p}, 1)$ and a sequence $(i(n),\ n)$, where $i(n) = [np] + 1$. We have verified that

$$n[E\eta_{(i,n)} - E\eta_{(i-1,n)}] \to \frac{1}{g(G^{-1}(p))}$$

as $n \to \infty$, uniformly in p for $p \in (\tilde{p}, 1)$. Finally, by the Glivenko-Cantelli theorem, and as noted in the proof of lemma 1, $F_n(\varepsilon + \Delta)$ converges to the Dirac measure on $F(\varepsilon + \Delta)$. Thus

$$\phi'_{n+1,n}(\bar{\varepsilon} + \Delta) \to f(\varepsilon + \Delta)/g(G^{-1}(F(\bar{\varepsilon} + \Delta)))$$

uniformly in Δ for $\Delta \in [-d, 0]$.

Finally, given the results in lemma 6, in particular (A31) and (A32), because $\phi'_n(\bar{\varepsilon} + \Delta)$ converges uniformly in Δ to $\phi'(\bar{\varepsilon} + \Delta)$, then $B''_n(\Delta, x_G) \to B''(\Delta, x_G)$ uniformly in $(\Delta,\ x_G)$ for $\Delta \in [-d, 0)$ as $n \to \infty$. QED

Proof of Theorem 5

Lemma 5 shows that for n large enough, there exists a profile (x^*_{Bn}, x^*_{Gn}) that satisfies the first-order conditions. We now show that no deviation from x^*_{Bn} is profitable for n sufficiently large. With additive quality, $\bar{\Delta} = \bar{\varepsilon} - \underline{\varepsilon}$ and $\underline{\Delta} = \max\{-\bar{\Delta}, \bar{x}_B - x_B\}$, and we may focus on deviations $\Delta \in [\underline{\Delta}, \bar{\Delta}]$. In the proof of theorem 1, we established that under assumptions 1 and 2, there exists $d > 0$ such that $B''(\Delta, x_G) < 0$ if $\Delta \in [-d, 0)$ and that $B''(\Delta, x_G) < 0$ for $\Delta \in (0, \bar{\Delta})$. Recall from the proof of lemma 6 that for $\Delta > 0$, $B''_n(\Delta, x_G) < \tilde{B}''_n(\Delta, x_G)$ and that $B''_n(\Delta, x_G)$ converges to $B''(\Delta,\ x_G)$ uniformly in $(\Delta,\ x_G)$; and for $\Delta < 0$, $B''_n(\Delta, x_G)$ converges to $B''(\Delta,\ x_G)$ uniformly in (Δ, x_G). Thus there exists N: $n > N$ such that, for Δ either in $[-b, 0)$ or in $(0, \bar{\Delta})$, $B''_n(\Delta, x^*_{Gn}) < \gamma/2$, where γ is the lower bound on $c''_B(\cdot)$. Thus $B''_n(\Delta, x^*_{Gn}) - c''_B(x^*_{Bn} + \Delta) < \gamma/2$ if either $\Delta \in [-b, 0)$ or $\Delta \in (0, \bar{\Delta})$, so that no such Δ deviation is profitable. In the proof of theorem 1, we also established that if $\bar{u}$ is small enough, there exists $d' \in (0,\ d)$ such that $B'(\Delta,\ x_G) \geq B'(0,\ x_G)$ for $\Delta \in [\underline{\Delta}, -d']$. Thus the payoff loss from a $-d'$ deviation is some $L > 0$ (since the payoff function is strictly concave on the interval $[-d', 0)$), and the payoff loss from any larger downward deviation is no less than L. Let $\bar{u}$ be sufficiently small such that a larger downward deviation, which results in being unmatched with probability one-half, leads to an expected payoff loss that is also greater than L. Since $(x^*_{Bn}, x^*_{Gn}) \to (x^*_B, x^*_G)$ and $B_n(\Delta, x_G)$ converges to $B(\Delta, x_G)$ uniformly in (Δ, x_G) as $n \to \infty$, there exists

$$N : n > N \Rightarrow [B_n(0, x^*_{Gn}) - c_B(x^*_{Bn})] - [B_n(\Delta, x^*_{Gn}) - c_B(x^*_{Bn} + \Delta)] > 0$$

for any $\Delta < -d'$. This completes the proof of theorem 5. QED

Theorem 5 shows that the equilibrium of the continuum model is the limit of a sequence of finite equilibria. We now provide a partial converse: the limit of a sequence of finite equilibria must be an equilibrium of the continuum model, if the payoff functions converge uniformly, so that the equilibrium correspondence is upper hemicontinuous.

Proposition 4. In the continuum agent model, let $U_B(\Delta | x_B, x_G)$ be the payoff function for any boy who chooses $x_B + \Delta$ when all other boys choose x_B and when all girls choose x_G and similarly for $U_G(\Delta | x_B, x_G)$ for a girl. And in the finite agent model with $2n + 1$ agents, let $U_{Bn}(\Delta | x_{Bn}, x_{Gn})$ and $U_{Gn}(\Delta | x_{Bn}, x_{Gn})$ be the analogous payoff functions. Suppose that, for $i = B$ and G, U_{in} converges uniformly in all three variables (Δ, x_B, x_G) to U_i. If for each n (x^*_{Bn}, x^*_{Gn}) is an equilibrium of the finite agent model and $\lim_{n\to\infty}(x^*_{Bn}, x^*_{Gn}) = (x^*_B, x^*_G)$, then (x^*_B, x^*_G) is an equilibrium of the continuum agent model.

Proof. Suppose not, so that $(x^*_B, x^*_G) = \lim_{n\to\infty}(x^*_{Bn}, x^*_{Gn})$ is not an equilibrium for the continuum agent model. Thus a boy (or girl—the argument is identical) must have a profitable deviation, that is, there exists $\Delta : U_B(\Delta | x^*_B, x^*_G) - U_B(0 | x^*_B, x^*_G) = 2\epsilon > 0$. Since $U_{Bn}(\cdot)$ converges to $U_B(\cdot)$ uniformly in all three arguments and since $(x^*_{Bn}, x^*_{Gn}) \to (x^*_B, x^*_G)$ as $n \to \infty$, there exists

$$N : n > N \Rightarrow |U_{Bn}(\Delta | x^*_{Bn}, x^*_{Gn}) - U_B(\Delta | x^*_B, x^*_G)| < \epsilon$$

and

$$|U_{Bn}(0 | x^*_{Bn}, x^*_{Gn}) - U_B(0 | x^*_B, x^*_G)| < \epsilon.$$

Thus,

$$\begin{aligned} &U_{Bn}(\Delta | x^*_{Bn}, x^*_{Gn}) - U_{Bn}(0 | x^*_{Bn}, x^*_{Gn}) \\ &> [U_B(\Delta | x^*_B, x^*_G) - \epsilon] - [U_B(0 | x^*_B, x^*_G) + \epsilon] > 0. \end{aligned}$$

Hence, for n sufficiently large, (x^*_{Bn}, x^*_{Gn}) is not an equilibrium of the finite model. QED

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