Solutions to Problem Set 4

Microeconomics I

Exercise 1: Consider the infinitely repeated game $G(\infty, \delta)$ based on the stage game below. Use the principle of optimality to find the set of discount factors for which the following strategy profile is a subgame perfect equilibrium:
1) in period 1 Player i plays a_i;
2) in every period after period 1, Player i plays b_i if (b_1, b_2) or (c_1, c_2) was played in the previous period; and
3) in every period after period 1, Player i plays c_i if (b_1, b_2) or (c_1, c_2) was not played in the previous period.

		Player 2		
		a_2	b_2	c_2
Player 1	a_1	4,4	3,2	1,1
	b_1	2,3	2,2	1,1
	c_1	1,1	1,1	-1,-1

If players stick to these strategies, then the action profiles played in periods 1,2,3,.... are

$$(a_1, a_2), (c_1, c_2), (b_1, b_2), (b_1, b_2), (b_1, b_2), \cdots$$

with (b_1, b_2) being played until infinity. Thus, the payoff from this strategy to each player is equal to

$$4 + \delta(-1) + \delta^2 \frac{2}{1-\delta}.$$

Since the game is symmetric, we will only analyze possible deviations from the perspective of the row player, as the analysis is the same for the column player.

Let us consider all of the possible one-shot deviations. Notice that his strategy prescribes actions only depending on the action profile played in the previous period. Therefore, on the path of play prescribed by the strategy, we only need to distinguish between three types of histories: the initial period ($t = 1$), the second period ($t = 2$), and all of the remaining periods ($t \geq 3$).
at $t = 1$: There are two possible deviations. Deviating to b_1 and then reverting back to the given strategy gives $2 + \delta(-1) + \delta^2 \frac{2}{1-\delta}$. Deviating to c_1 and then reverting back to the given strategy gives $1 + \delta(-1) + \delta^2 \frac{2}{1-\delta}$. Clearly, b_1 gives the higher payoff from deviating, so in order to ensure that the strategy is an equilibrium, this deviation should not be profitable:

$$4 + \delta(-1) + \delta^2 \frac{2}{1-\delta} \geq 2 + \delta(-1) + \delta^2 \frac{2}{1-\delta} \tag{1}$$

This holds for any $\delta \in (0, 1)$ so we do not have profitable deviations at $t = 1$.
at $t = 2$: There are two possible deviations – to a_1 or to b_1 – but they both give the same payoff: $1 + \delta(-1) + \delta^2 \frac{2}{1-\delta}$. This deviation would not be profitable if

$$-1 + \delta \frac{2}{1-\delta} \geq 1 + \delta(-1) + \delta^2 \frac{2}{1-\delta}. \tag{2}$$

The above equation can be rearranged as

$$\delta + \delta\frac{2}{1-\delta}(1-\delta) \geq 2 \Leftrightarrow \delta \geq \frac{2}{3} \tag{3}$$

So, we need $\delta \geq \frac{2}{3}$ in order not to have profitable deviations at $t = 2$.
at $t \geq 3$: There are two possible deviations. Deviating to a_1 and then reverting back to the given strategy gives $3 + \delta(-1) + \delta^2\frac{2}{1-\delta}$. Deviating to c_1 and then reverting back to the given strategy gives $1 + \delta(-1) + \delta^2\frac{2}{1-\delta}$. Clearly, a_1 gives the higher payoff from deviating, so in order to ensure that the strategy is an equilibrium, this deviation should not be profitable:

$$\frac{2}{1-\delta} \geq 3 + \delta(-1) + \delta^2\frac{2}{1-\delta}. \tag{4}$$

The above equation can be rearranged to

$$\frac{2}{1-\delta}(1-\delta^2) \geq 3 - \delta \tag{5}$$

which in turn gives

$$2(1+\delta) \geq 3 - \delta \Leftrightarrow \delta \geq \frac{1}{3}. \tag{6}$$

So, we need $\delta \geq \frac{1}{3}$ in order not to have profitable deviations at $t \geq 3$.

Thus far we have considered only deviations at histories on the path of play, that is if both players have followed the strategy. Let us now think about deviations at histories where previous deviations have happened. There are two relevant classes of histories where deviations have happened previously:
1) either a deviation occurred one period ago, in which case we are supposed to play (c_1, c_2) in the current period
2) or a deviation occurred two periods ago, in which case we are supposed to play (b_1, b_2) in the current period.
But then, the deviations at histories of type 1) are equivalent to the deviations at $t = 2$, while the deviations at histories of type 2) are equivalent to the deviations at $t \geq 3$. Therefore, the conditions we need to ensure no deviations are profitable at these histories are also $\delta \geq \frac{2}{3}$ and $\delta \geq \frac{1}{3}$ respectively.

In conclusion, the binding inequality is $\delta \geq \frac{2}{3}$ and so the set of discount factors for which the given strategy is a SPE is $\delta \in [\frac{2}{3}, 1]$.

Exercise 2: (Repeated Prisoner's Dilemma) Consider the following stage game G:

		Player 2	
		C	D
Player 1	C	-1,-1	-4,0
	D	0,-4	-3,-3

Show that for high enough δ there is a subgame perfect equilibrium (SPE) σ of the infinitely repeated game $G(\infty, \delta)$ for which $u_1(\sigma) = u_2(\sigma) = -1$.

Consider the following strategy for each player:
1) in period 1 Player i plays C;
2) in every period after period 1, Player i plays C if (C, C) has been played in all previous periods; and
3) in every period after period 1, Player i plays D otherwise.

Let us check whether the above strategy profile is a SPE. There are two types of histories:

- where no deviations have previously occurred: in this case sticking to the strategy and playing C gives a payoff of -1 forever, so $\frac{(-1)}{1-\delta}$. One shot deviation to D and then reverting back to the strategy gives $0 + \delta\frac{(-3)}{1-\delta}$. This deviation is not profitable if

$$-\frac{1}{1-\delta} \geq 0 - \delta\frac{3}{1-\delta} \Leftrightarrow \delta \geq \frac{1}{3}$$

- where a deviation has occurred previously: in this case sticking to the strategy and playing D gives a payoff of -3 forever, so $\frac{(-3)}{1-\delta}$. One shot deviation to C and then reverting back to the strategy gives $-4 + \delta\frac{(-3)}{1-\delta}$. This deviation is not profitable if

$$-\frac{3}{1-\delta} \geq -4 - \delta\frac{3}{1-\delta}$$

  which holds for any $\delta \in (0, 1)$.

Hence, the condition which needs to hold for this strategy profile to be a SPE is $\delta \geq \frac{1}{3}$. In this equilibrium, (C, C) is played in every period and so each player gets a payoff of -1 every period i.e. $\frac{(-1)}{1-\delta}$ in total. When normalized (multiplied by $(1-\delta)$) this gives an average payoff of -1 per period.

Exercise 3: Consider the following stage game G (G is a game of "Chicken"):

		Player 2 S	Player 2 KG
Player 1	S	4,4	1,6
	KG	6,1	0,0

where S stands for "Swerve" and KG stands for "Keep Going".

(a) Find every Nash equilibrium of G.
The NE of the stage game G are (S, KG), (KG, S), and $(\sigma_1(S) = \frac{1}{3}, \sigma_2(S) = \frac{1}{3})$.

(b) Find the strategy m_1 of Player 2 that minmaxes Player 1, and the corresponding minmax value $\underline{v}_1$ of Player 1. (G is symmetric, so that is also Player 2's minmax value.)
Let p denote the probability with which Player 2 plays S. Then the best-response payoff to Player 1 is $w_1(p) = \max\{1 + 3p, 6p\}$. This is minimized at $p = 0$, i.e. $m_1 = KG$. That is, by playing KG Player 2 minimizes the value of Player 1's best response payoff $w_1(p)$. and we get $\min_p w_1(p) = \underline{v}_1 = 1$.

(c) Carefully draw the utility possibility set, and the set of payoffs identified in the Fudenberg and Maskin Folk Theorem as SPE payoffs of $G(\infty, \delta)$ for high enough δ.
See attached graph.

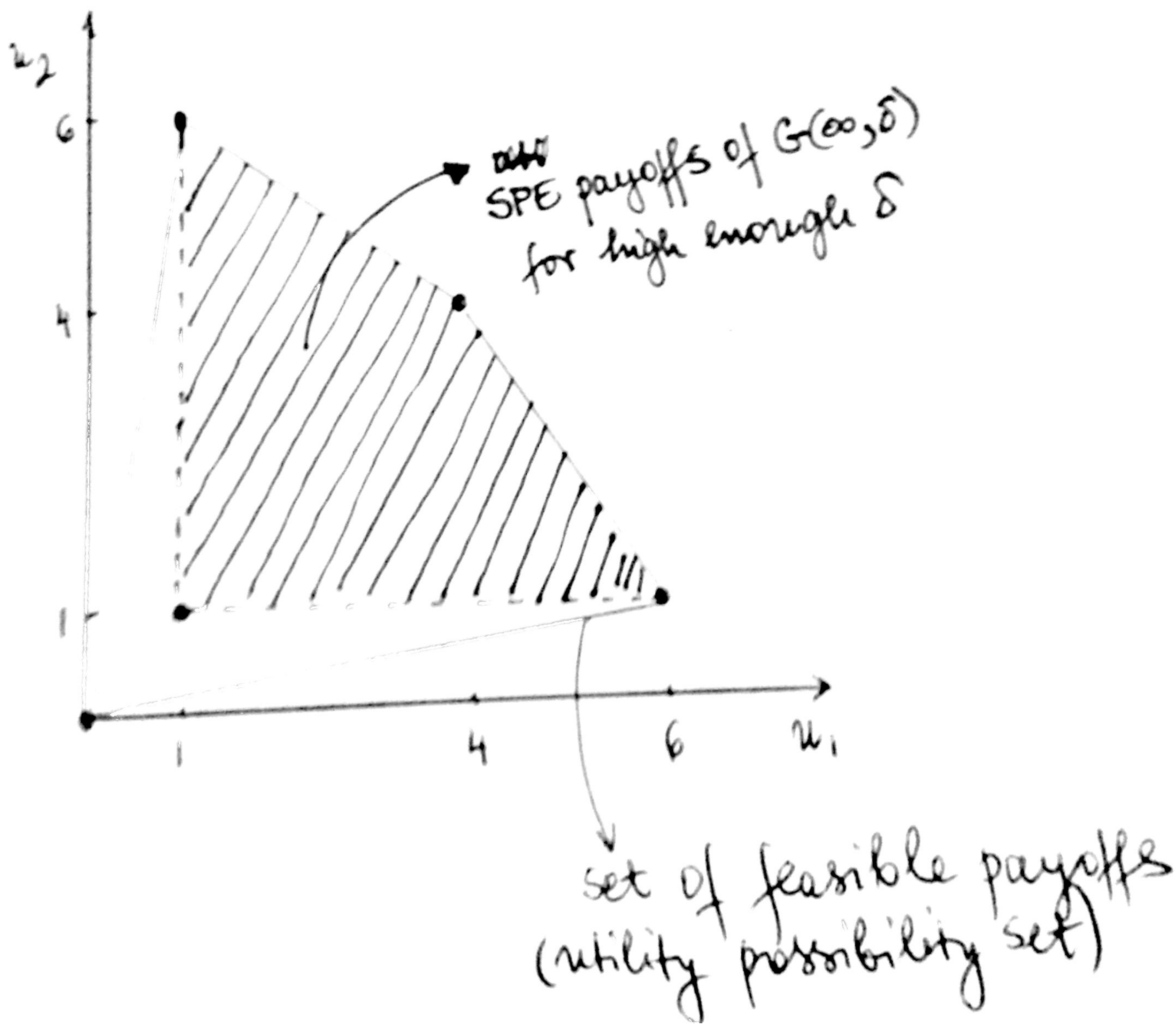
(d) For what values of δ is $((S, S), (S, S), (S, S), \dots)$ a Nash equilibrium outcome of $G(\infty, \delta)$? NOTE: The equilibrium need not be subgame perfect.

On the equilibrium path, each player gets a payoff of 4 in every period, i.e. $\frac{4}{1-\delta}$. The harshest possible punishment that can be inflicted on any player is $\underline{v}_i = 1$ in every period after a deviation has occurred. Hence, deviating to

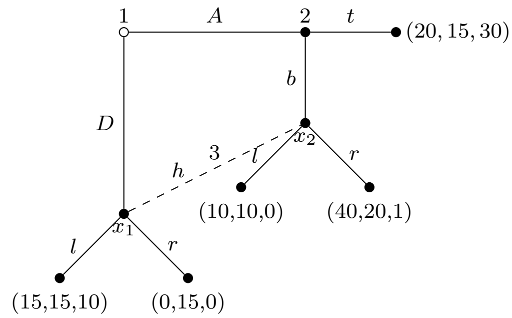
KG in any period and receiving the minmax payoff from then onwards gives $6 + \delta \frac{1}{1-\delta}$. Therefore, deviating is not profitable for

$$\frac{4}{1-\delta} \geq 6 + \delta \frac{1}{1-\delta} \Leftrightarrow \delta \geq \frac{2}{5}.$$

Hence, for any $\delta \geq \frac{2}{5}$, $((S,S),(S,S),(S,S),\dots)$ can be supported as a NE outcome of $G(\infty,\delta)$. For example, the following strategy for each player supports this as a NE:
1) in period 1 Player i plays S;
2) in every period after period 1, Player i plays S if (S,S) has been played in all previous periods; and
3) in every period after period 1, Player i plays KG otherwise.
So for $\delta \geq \frac{2}{5}$ the above strategy profile is a NE of $G(\infty,\delta)$. However, it is not a SPE.


u_2
6
4
1
1
4
6
u_1
SPE payoffs of $G(\infty, \delta)$ for high enough δ
set of feasible payoffs (utility possibility set)

Exercise 4: Consider the following extensive form game:



(a) Write the corresponding normal form.

Player 3: l

Player 1 \ Player 2	b	t
D	15,15,10	15,15,10
A	10,10,0	20,15,30

Player 3: r

Player 1 \ Player 2	b	t
D	0,15,0	0,15,0
A	40,20,1	20,15,30

(b) Show that no player has any weakly dominated strategies.
$u_1(D|bl) = 15 > u_1(A|bl) = 10$
$u_1(D|tl) = 15 < u_1(A|tl) = 20$
$\Rightarrow$ neither A nor D is weakly dominated for Player 1.

$u_2(t|Al) = 15 > u_2(b|Al) = 10$
$u_2(t|Ar) = 15 < u_2(b|Ar) = 20$
$\Rightarrow$ neither b nor t is weakly dominated for Player 2.

$u_3(l|Db) = 10 > u_3(r|Db) = 0$
$u_3(l|Ab) = 0 < u_3(r|Ab) = 1$
$\Rightarrow$ neither l nor r is weakly dominated for Player 3.

(c) Let p be the probability that Player 3 chooses action l. For what values of p is (A, t, p) a Nash equilibrium.

Notice that l and r are both best responses to (A, t) for Player 3, and therefore he is willing to mix with any probability. It remains to be checked for which values of p A is a best response to $(t, \sigma(l) = p)$ for Player 1 and t is a best response to $(A, \sigma(l) = p)$ for Player 2.
For Player 1, the expected payoff from playing A given $(t, \sigma(l) = p)$ is

$$Eu_1(A|t, \sigma(l) = p) = 20p + 20(1 - p) = 20$$

and the expected payoff from deviating to D

$$Eu_1(D|t, \sigma(l) = p) = 15p + 0(1 - p) = 15p < 20$$

So A is a best response to $(t, \sigma(l) = p)$.
For Player 2, the expected payoff from playing t given $(A, \sigma(l) = p)$ is

$$Eu_2(t|A, \sigma(l) = p) = 15p + 15(1 - p) = 15$$

and the expected payoff from deviating to b given $(A, \sigma(l) = p)$ is

$$Eu_2(b|A, \sigma(l) = p) = 10p + 20(1 - p) = 20 - 10p.$$

Therefore, t is a best response to $(A, \sigma(l) = p)$ if $15 \geq 20 - 10p \Leftrightarrow p \geq \frac{1}{2}$.
Hence, $(A, t, \sigma(l) = p)$ is a Nash equilibrium for all $p \in [\frac{1}{2}, 1]$.

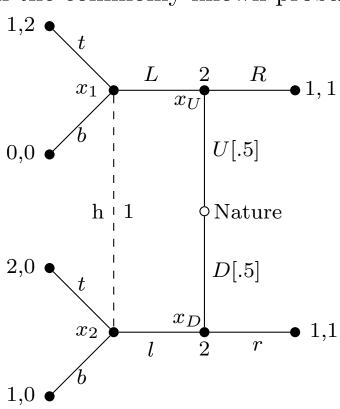
(d) Show that whenever (A, t, p) is a Nash equilibrium, then it is also part of a perfect Bayesian equilibrium.
For $p \in [\frac{1}{2}, 1]$, $(A, t, \sigma(l) = p)$ is a NE. It is also subgame perfect equilibrium since the whole game is the only subgame.

Let us denote the beliefs of Player 3 by $\mu_3(x_1|h)$. For Player 3 to be mixing between l and r at h, he needs to be indifferent between them:

$$10\mu_3(x_1|h) + 0(1 - \mu_3(x_1|h)) = 0\mu_3(x_1|h) + 1(1 - \mu_3(x_1|h))$$

which gives $\mu_3(x_1|h) = \frac{1}{11}$. Hence, $[(A, t, \sigma(l) = p), \mu_3(x_1|h) = \frac{1}{11}]$ is a perfect Bayesian equilibrium for any $p \in [\frac{1}{2}, 1]$.

Exercise 5: Consider the following extensive form game in which Nature moves first with the commonly known probabilities given in brackets:



The first payoff is Player 1's, and the second payoff is Player 2's.

(a) Find all the perfect Bayesian equilibria (PBE).

The corresponding normal form is:

Player 2

		Ll	Lr	Rl	Rr
Player 1	t	1.5,1	1,1.5	1.5,0.5	1,1
	b	0.5,0	0.5,0.5	1,0.5	1,1

The pure strategy NE are (t, Lr) and (b, Rr). There is a continuum of mixed strategy NE $(\sigma_1(t) \in (0, 0.5], Rr)$.
All of the NE are also SPE since this game has no proper subgames.
There is a unique PBE given by $[(t, Lr), \mu_1(x_1) = 1]$.

(b) Find a Nash equilibrium that is not a part of a PBE.
(b, Rr) is not part of a PBE, since there are no beliefs at h that would make Player 1 choose b instead of t – notice that conditional on being at information set h, b is strictly dominated by t.
Similarly, $(\sigma_1(t) \in (0, 0.5], Rr)$ is not a part of PBE, since there are no beliefs at h that would make Player 1 indifferent between b and t.