Solutions to Problem Set 3

Core Micro

<u>Exercise 1:</u> Consider a first-price sealed-bid auction of an object with two bidders. Each bidder i's valuation of the object is v_i, which is known to both bidders. The auction rules are that each player submits a bid in a sealed envelope. The envelopes are then opened, and the bidder who has submitted the highest bid gets the object and pays the auctioneer the amount of his bid. If the bidders submit the same bid, each gets the object with probability $\frac{1}{2}$. Bids must be in dollar multiples (assume that valuations are also).

(a) Are any strategies strictly dominated?
This is a simultaneous move game where the actions are the bides. Denote by $b_i \in \mathbb{R}_+$ bidder i's bid. The payoffs for bidder i are given by

$$u_i(b_i, b_j) = \begin{cases} v_i - b_i & \text{if } b_i > b_j \\ \frac{1}{2}(v_i - b_i) & \text{if } b_i = b_j \\ 0 & \text{if } b_i < b_j \end{cases}$$

No strategy is strictly dominated. To prove this consider b'_i which strictly dominates b_i. Then, it must be that $u_i(b'_i, b_j) > u_i(b_i, b_j)$ for all b_j. Consider $b^*_j = \max\{b_i, b'_i\} + 1$. Then, bidder j wins the auction both if agent i bids b_i or if he bids b'_i. So, we have that $u_i(b_i, b^*_j) = u_i(b'_i, b^*_j) = 0$. This is a contradiction to b_i being strictly dominated.

(b) Are any strategies weakly dominated?
All strategies $b_i > v_i$ are weakly dominated by $b'_i = v_i$. To show this consider the payoffs in all the possible scenarios:

(i) if $b_j < v_i$ then $u_i(b'_i, b_j) = 0 > v_i - b_i = u_i(b_i, b_j)$
(ii) if $b_j = v_i$ then $u_i(b'_i, b_j) = \frac{1}{2}(v_i - v_i) = 0 > v_i - b_i = u_i(b_i, b_j)$
(iii) if $v_i < b_j < b_i$ then $u_i(b'_i, b_j) = 0 > v_i - b_i = u_i(b_i, b_j)$
(iv) if $v_i < b_j = b_i$ then $u_i(b'_i, b_j) = 0 > \frac{1}{2}(v_i - b_i) = u_i(b_i, b_j)$
(v) if $b_j > b_i$ then $u_i(b'_i, b_j) = 0 = u_i(b_i, b_j)$

Thus, $u_i(b'_i, b_j) \geq u_i(b_i, b_j)$ for all b_j and $u_i(b'_i, b_j) > u_i(b_i, b_j)$ for some b_j. Therefore, any $b_i > v_i$ is weakly dominated by $b'_i = v_i$.

There are more weakly dominated strategies (due to the bids and valuations being only in dollar multiples). For example , it is easy to show that when $v_i > 2$, then $b_i = 0$ is weakly dominated by $b'_i = 1$. And when $v_i \in \{1, 2\}$, then $b_i = v_i$ is weakly dominated by $b'_i = v_i - 1$.

(c) Is there a Nash equilibrium? If so, what is it? Is it unique?
We first obtain the best response correspondence $R_i(b_j)$ for each player i —

the set of actions b_i which maximize $u_i(b_i, b_j)$ given b_j. These are given by:

$$R_i(b_j) = \begin{cases} b_j + 1 & \text{if } v_i - 2 > b_j \\ \{b_j, b_j + 1\} & \text{if } v_i - 2 = b_j \\ b_j & \text{if } v_i - 1 = b_j \\ \{0, 1, 2, \ldots, v_i\} & \text{if } v_i = b_j \\ \{0, 1, 2, \ldots, b_j - 1\} & \text{if } v_i < b_j \end{cases}$$

A NE is a pair (b_i^*, b_j^*) where $b_i^* \in R_i(b_j^*)$ and $b_j^* \in R_j(b_i^*)$. By looking at the above best response correspondences, it can be verified that a NE always exists and for most parameter specifications is not unique. For example, when $v_i \geq v_j$, the NE are as follows:

(i) if $v_i = v_j$ then (v_i, v_j), $(v_i - 1, v_j - 1)$, $(v_i - 2, v_j - 2)$ are NE

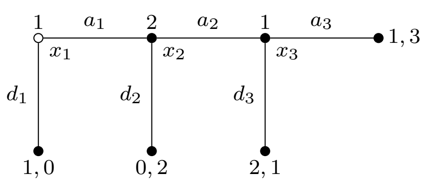
(ii) if $v_i = v_j + 1$ then $(v_i - 1, v_i - 1)$, $(v_i - 1, v_i - 2)$, $(v_i - 2, v_i - 2)$ are NE

(iii) if $v_i = v_j + 2$ then $(v_i - 2, v_i - 2)$ is a NE

(iv) if $v_i > v_j + 1$ then $(v_i - x, v_i - x - 1)$ with $1 \leq x \leq v_i - v_j$ are NE

The purpose of this exercise is for you to formulate the best response correspondences correctly and to engage with attempting to find NE depending on the relationship between the parameters v_i and v_j. Even if you don't succeed in finding all NE, the goal is to learn how to think about best responses to best responses and to find at least some equilibria based on this.

<u>Exercise 2:</u> Consider the following extensive form game:



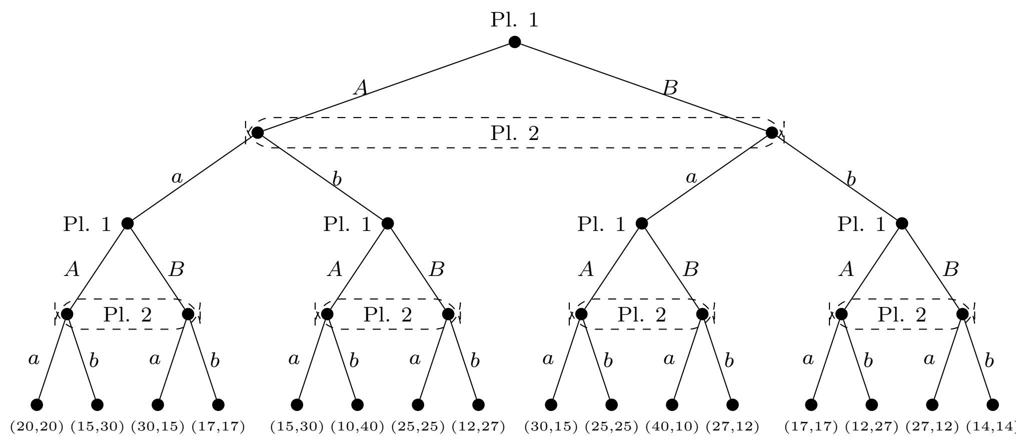
(a) Find all the subgame perfect Nash equilibria.
SPNE: $((d_1, d_3), d_2)$.

(b) Find all the Nash equilibria.
NE: $((d_1, a_3); d_2)$; $((d_1, d_3), d_2)$.

<u>Exercise 3:</u> The following normal form game is played twice:

		Player 2	
		a	b
Player 1	A	10,10	5,20
	B	20,5	7,7

(a) Draw the extensive form game. How many pure strategies does each player have?



Each player has $2 \cdot 2^4 = 32$ possible pure strategies.

(b) Find all the subgame perfect Nash equilibria.
The only SPNE is $\big((B, B|Aa, B|Ab, B|Ba, B|Bb), (b, b|Aa, b|Ab, b|Ba, b|Bb)\big)$: both players play their respective dominant strategies at every information set and at each stage. The SPNE outcome is $\big((B, b), (B, b)\big)$.

Exercise 4: A Stackelberg duopoly has two firms — firm 1 and firm 2 — with firm 1 choosing output first and firm 2 choosing output second, after observing the choice of firm 1. Suppose that the inverse demand function is $P(Q) = 6 - Q$, where $Q = q_1 + q_2$ is aggregate output. Each firm has constant marginal cost of £4 per unit, and a capacity constraint of 3 units.

(a) Define formally the strategy set of each firm. (*Hint:* Firm 2's strategy is a function.)
$q_1 \in [0, 3]$
$q_2 : [0, 3] \to [0, 3]$

(b) Find a Nash equilibrium in which the Cournot outputs are produced.
First, we derive the Cournot output quantities. In the simultaneous move game, firm i chooses q_i to maximize its profit $\pi_i(q_i, q_{-i})$:

$$\max_{q_i \in [0,3]} \quad \pi_i(q_i, q_{-i}) = (6 - q_i - q_{-i})q_i - 4q_i = (2 - q_i - q_{-i})q_i$$

FOC: $\frac{\partial \pi_i}{\partial q_i} = 2 - 2q_i - q_{-i} = 0$
Hence, we obtain the best response of firm i as a function of the output of its opponent as:

$$q_i^{*C}(q_{-i}) = 1 - 0.5q_{-i}$$

Substituting in with $q_{-i}^{*C}(q_i) = 1 - 0.5q_i$ and solving for q_i^{*C} we obtain the Cournot equilibrium quantities $q^{*C} = q_i^{*C} = q_{-i}^{*C} = \frac{2}{3}$ and payoffs $\pi_i^{*C} = \pi_{-i}^{*C} = \frac{4}{9}$.

Second, notice that for any choice q_1 of firm 1, the best response of firm 2 in the Stackelberg duopoly game is given by $q_2^{*S}(q_1) = 1 - 0.5q_1$.

Now consider the following strategies:
$q_1 = \frac{2}{3}$

$q_2(q_1) = \begin{cases} 1 - 0.5q_1 & \text{if } q_1 = 2/3 \\ 3 & \text{otherwise} \end{cases}$

It can be easily checked that both firms are best responding to the other one's strategy, so this pair of strategies constitutes a NE in which the Cournot quantities $q_1 = q_2 = q^{*C} = \frac{2}{3}$ are produced.

(c) Find a Nash equilibrium in which firm 2 produces the monopoly output and firm 1 produces nothing.
The monopoly quantity is derived as

$$\max_{q\in[0,3]} \quad \pi(q) = (6-q)q - 4q = (2-q)q$$

FOC: $\frac{\partial \pi}{\partial q} = 2 - 2q = 0$
Hence, we obtain the monopoly quantity $q^{*M} = 1$. Now consider the following strategies:
$q_1 = 0$
$q_2(q_1) = \begin{cases} 1 - 0.5q_1 & \text{if } q_1 = 0 \\ 3 & \text{otherwise} \end{cases}$
It can be easily checked that both firms are best responding to the other one's strategy, so this pair of strategies constitutes a NE in which firm 2 produces the monopoly quantity $q_2 = q^{*M} = 1$ and firm 1 produces nothing.

(d) Find the subgame perfect Nash equilibria.
In any SPE, firm 2 must be playing a best response in any subgame, i.e. both on and off the equilibrium path. Hence, for any q_1, firm 2 must respond with $q_2^{*S}(q_1) = 1 - 0.5q_1$. Firm 1 knows that firm 2 will best respond in such a way for any q_1 and takes that into account when choosing its optimal action:

$$\max_{q_1\in[0,3]} \quad \pi_1(q_1, q_2) = (6 - q_1 - q_2^{*S}(q_1))q_1 - 4q_1 = (1 - 0.5q_1)q_1$$

FOC: $\frac{\partial \pi_1}{\partial q_1} = 1 - q_1 = 0$
Hence, we obtain $q_1^{*S} = 1$. The best response of firm 2 to this is $q_2^{*S}(1) = 1 - 0.5 \cdot 1$. Hence, the SPE is given by
$q_1^{*S} = 1$
$q_2^{*S}(q_1) = \begin{cases} 1 - 0.5q_1 & \text{if } q_1 \in [0,2] \\ 0 & \text{otherwise} \end{cases}$
The equilibrium quantities are thus given by $q_1^{*S} = 1$ and $q_2^{*S} = 0.5$.

Exercise 5: Suppose n players use an ultimatum procedure to share an apple pie. First, player 1 proposes a division. Then the others simultaneously respond "yes" or "no." If they all say "yes" , the proposed division is implemented. Otherwise, the pie is fed to Penny the dog. Each player prefers more pie to less, and is indifferent about how much pie any other player or dog consumes.

(a) Define formally the strategy set of each player.
$A_1 = \{p \in \mathbb{R}^n_+ : \sum_{i=1}^{n} p_i = 1\}$
$A_i : \{p \in \mathbb{R}^n_+ : \sum_{i=1}^{n} p_i = 1\} \rightarrow \{\text{Yes}, \text{No}\}$ for each $i \geq 2$

(b) Find the subgame perfect Nash equilibria when $n = 2$ and $n = 3$.

For $n = 2$ the SPNE is $\big((p_1 = 1, p_2 = 0), (\text{Yes}|p_2 \geq 0)\big)$.
Notice that $\big((p_1 = 1 - \varepsilon, p_2 = \varepsilon), (\text{Yes}|p_2 > 0, \text{No}|p_2 = 0)\big)$ would also be a SPNE, but since Player 1 would like to make ε as small as possible and since there is no minimal amount of pie, this is not well-defined.

For $n = 3$ there is a multiplicity of SPNE, which can be grouped as follows:

$\big((p_1 = 1, p_2 = 0, p_3 = 0), (\text{Yes}|p_2 \geq 0), (\text{Yes}|p_3 \geq 0)\big)$

$\big((p_1 = 1-q-r, p_2 = q, p_3 = r), (\text{Yes}|p_2 \geq q \,\text{and}\, p_3 \geq r, \text{No}|p_2 < q \,\text{or}\, p_3 < r), (\text{Yes}|p_2 \geq q \,\text{and}\, p_3 \geq r, \text{No}|p_2 < q \,\text{or}\, p_3 < r)\big)$ for any $p + q \leq 1$

$\big((p_1 = 1 - S, p_2 = q, p_3 = S - q), (\text{Yes}|p_2 + p_3 \geq S, \text{No}|p_2 + p_3 < S), (\text{Yes}|p_2 + p_3 \geq S, \text{No}|p_2 + p_3 < S)\big)$ for any $q \leq S \leq 1$

Notice that
$\big((p_1 = 1 - \varepsilon, p_2 = \varepsilon, p_3 = 0), (\text{Yes}|p_2 > 0, \text{No}|p_2 = 0), (\text{Yes}|p_3 \geq 0)\big)$,
$\big((p_1 = 1 - \varepsilon, p_2 = 0, p_3 = \varepsilon), (\text{Yes}|p_2 \geq 0), (\text{Yes}|p_3 > 0, \text{No}|p_3 = 0)\big)$, and
$\big((p_1 = 1 - \varepsilon, p_2 = \varepsilon, p_3 = \varepsilon), (\text{Yes}|p_2 > 0, \text{No}|p_2 = 0), (\text{Yes}|p_3 > 0, \text{No}|p_3 = 0)\big)$
would also be SPNE, but since Player 1 would like to make ε as small as possible and since there is no minimal amount of pie, this is not well-defined.

<u>Exercise 6:</u> Consider the following Pirate Game: There are R pirates who must decide how to divide 100 gold pieces among themselves. The gold pieces are indivisible, so a division is feasible only if each pirate gets a whole number of gold pieces. The mechanism they use is as follows: Pirate 1 proposes a division. Then Pirate 2 can accept or reject it. If he accepts, the proposed division is implemented and the game is over. If he rejects, Pirate 1 is thrown to the sharks, and then Pirate 2 proposes a division to Pirate 3, and so on. If pirate R rejects Pirate $R-1$'s proposal, then Pirate R gets all 100 gold pieces. Pirates prefer more gold to less and are indifferent about about how much gold any other pirate gets; being fed to the sharks is their least favorite thing. Watching another pirate being fed to the sharks gives a pirate positive utility, but a pirate always prefers an extra gold piece to watching sharks eat. Describe the subgame perfect Nash equilibria of this game. How does your answer depend on the value of R?

Since this is a game of perfect information, the subgame perfect Nash equilibria are the backwards induction solutions.

Suppose $R = 2$. Pirate 2 always rejects Pirate 1's offer, no matter what it is. That way, Pirate 2 gets both the 100 gold pieces and the utility from watching Pirate 1 being fed to the sharks. Hence, the set of SPE is characterized by Pirate 1 proposing any division and Pirate 2 rejecting any proposed division. The payoffs are $(-\infty, 100 + c)$, where c is the utility gained from watching another pirate being fed to the sharks.

Suppose $R = 3$. Pirate 3 always rejects Pirate 2's offer, no matter what it is. That way, Pirate 3 gets both the 100 gold pieces and the utility from watching Pirate 2 being fed to the sharks. Pirate 2 knows that if it ever comes to him to make an offer, it will be rejected and he will die. Hence, Pirate 2 would accept any offer so that the game would end and he would avoid being fed to the sharks by Pirate 3. Pirate 1 knows that Pirate 2 would accept any offer in order to avoid being fed to the sharks and so proposes a "division" where he keeps all the 100 gold pieced and gives Pirate 2 and Pirate 3 zero goldpieces. The payoffs are $(100, 0, 0)$.

Suppose $R = 4$. Pirate 4 always rejects Pirate 3's offer, no matter what it is. That way, Pirate 4 gets both the 100 gold pieces and the utility from watching Pirate 3 being fed to the sharks. Pirate 3 knows that if it ever comes to him to make an offer, it will be rejected and he will die. Hence, Pirate 3 would accept any offer so that the game would end and he would avoid being fed to the sharks by Pirate 4. Pirate 2 knows that Pirate 3 will accept any offer in order to avoid being fed to the sharks and so Pirate 2's life is guaranteed. Therefore, Pirate 2 would reject any offer Pirate 1 makes and would propose a "division" to Pirate 3, where he gets to keep all the gold pieces. The payoffs are $(-\infty, 100 + c, 0, 0)$.

And so on... The pattern shows that when R is **odd**, Pirate 1 will keep all the gold pieces for himself. So the SPE in this case is: Pirate 1 offers to keep all 100 pieces for himself, Pirate 2 always accepts, game over. The equilibrium payoffs are $(100, 0, 0, \ldots, 0)$. Denoting by $d_i \in \mathbb{R}_+^{R-i+1} = D_i$ the proposed division of the gold pieces by Pirate i, the equilibrium strategies, including at those subgames that are not reached in equilibrium, can be specified as:

$$\left\{ \begin{array}{ll} s_1 = (d_1 = (d_1^1 = 100, d_1^2 = 0, \ldots, d_1^R = 0)) & \\ s_{2k} = (\text{accept}, d_{2k} \in D_{2k}) & \text{for } k = 1, 2, \ldots, \frac{R-1}{2} \\ s_{2k+1} = (\text{reject}, d_{2k+1} = (d_{2k+1}^{2k+1} = 100, d_{2k+1}^{2k+2} = 0, \ldots d_{2k+1}^R = 0) & \text{for } k = 1, 2, \ldots, \frac{R-1}{2} \end{array} \right\}.$$

On the other hand, when R is **even**, Pirate 2 will keep all the gold pieces for himself. The SPE in this case is: Pirate 1 makes any offer, but Pirate 2 rejects it. Then Pirate 2 proposes to keep all the coins for himself and Pirate 3 accepts it. The equilibrium payoffs are $(-\infty, 100 + c, 0, 0, \ldots 0)$. The equilibrium strategies, including at those subgames that are not reached in equilibrium, can be specified as:

$$\left\{ \begin{array}{ll} s_1 = (d_1 \in D_1) & \\ s_{2k} = (\text{reject}, d_{2k} = (d_{2k}^{2k} = 100, d_{2k}^{2k+1} = 0, \ldots d_2^R = 0)) & \text{for } k = 1, 2, \ldots, \frac{R}{2} \\ s_{2k+1} = (\text{accept}, d_{2k+1} \in D_{2k+1}) & \text{for } k = 1, 2, \ldots, \frac{R}{2} \end{array} \right\}.$$