Solutions to Problem Set 2

Microeconomics I

Exercise 1: The game matrix below gives Player 1's payoffs:

		Player 2 L	Player 2 R
Player 1	U	x	0
	D	0	y

where $x > y > 0$. Let p be the probability with which Player 1 believes that Player 2 will play L. Derive the best response correspondence $BR(p)$.

Let us denote:
$u_1(U|p)$ = Player 1's expected utility from playing U, when he believes that Player 2 will play L with probability p
and
$u_1(D|p)$ = Player 1's expected utility from playing D, when he believes that Player 2 will play L with probability p.
Then, we have

$$u_1(U|p) = p \cdot x + (1-p) \cdot 0 = p \cdot x$$

and

$$u_1(D|p) = p \cdot 0 + (1-p) \cdot y = (1-p) \cdot y.$$

The point of indifference is where $u_1(U|p) = u_1(D|p)$, which happens for $p = \frac{y}{x+y}$. So the best response correspondence is

$$BR(p) = \arg \max_{U,D,\Delta(\{U,D\})} \{u_1(\cdot|p)\} = \begin{cases} D & \text{if } 0 \leq p \leq \frac{y}{x+y} \\ \Delta(\{U,D\}) & \text{if } p = \frac{y}{x+y} \\ U & \text{if } 1 \geq p \geq \frac{y}{x+y}. \end{cases}$$

where $\Delta(\{U,D\}) = \{(q, 1-q) : 1 \geq q \geq 0, \text{Prob}(U) = q, \text{Prob}(D) = 1-q\}$.

Note that in the above, we have included *mixed strategy* best responses. If we consider only *pure strategy* best responses , then the best response correspondence is given by

$$BR(p) = \arg \max_{U,D} \{u_1(\cdot|p)\} = \begin{cases} D & \text{if } 0 \leq p \leq \frac{y}{x+y} \\ \{U,D\} & \text{if } p = \frac{y}{x+y} \\ U & \text{if } 1 \geq p \geq \frac{y}{x+y}. \end{cases}$$

Exercise 2: The game matrix below gives Player 1's payoffs:

		Player 2	
		S	D
Player 1	U	15	90
	M	B	75
	D	55	40

Let q be the probability with which Player 1 believes that Player 2 will play S.

(a) Suppose that $B = 35$. Find the three ranges of values of q for which U, M and D are optimal, respectively (and draw a picture of expected utility versus q). Is any action strictly dominated, and if so, by what mixed action? (Draw another picture, utility when Player 2 plays S versus utility when Player 2 plays D.)
$u_1(U|q) = 15q + 90(1-q) = 90 - 75q$
$u_1(M|q) = 35q + 75(1-q) = 75 - 40q$
$u_1(D|q) = 55q + 40(1-q) = 40 + 15q$

$$\arg\max_{\{U,M,D\}} u_1(\cdot|q) = \begin{cases} U & \text{if } q \in [0, \frac{3}{7}] \\ M & \text{if } q \in [\frac{3}{7}, \frac{7}{11}] \\ D & \text{if } q \in [\frac{7}{11}, 1] \end{cases}$$

No action is strictly dominated (see attached graphs).

(b) Repeat (a), assuming now that $B = 20$.

$u_1(U|q) = 15q + 90(1-q) = 90 - 75q$
$u_1(M|q) = 20q + 75(1-q) = 75 - 55q$
$u_1(D|q) = 55q + 40(1-q) = 40 + 15q$

$$\arg\max_{\{U,M,D\}} u_1(\cdot|q) = \begin{cases} U & \text{if } q \in [0, \frac{5}{9}] \\ D & \text{if } q \in [\frac{5}{9}, 1] \end{cases}$$

M is strictly dominated by a set of mixed actions between U and D (see attached graphs). To find the set of these mixed actions, consider the mixed action σ_p with $\sigma_p(U) = p$ and $\sigma_p(D) = 1 - p$. Then, such a mixed action has the following payoffs given the pure actions of Player 2:

$$u_1(\sigma_p|S) = 15p + 55(1-p) = 55 - 40p$$

$$u_1(\sigma_p|D) = 90p + 40(1-p) = 40 + 50p.$$

Therefore, σ_p would strictly dominate M if

$$u_1(\sigma_p|S) = 55 - 40p > 20 = u_1(M|S)$$

and

$$u_1(\sigma_p|D) = 40 + 50p > 75 = u_1(M|D).$$

The first inequality requires $p < \frac{7}{8}$ while the second requires $p > \frac{7}{10}$. Therefore, for any $p \in \left(\frac{7}{10}, \frac{7}{8}\right)$, σ_p strictly dominates M.

(c) For what range of values of B is action M strictly dominated?
M will be dominated by σ_p with $\sigma_p(U) = p$ and $\sigma_p(D) = 1 - p$ if there exists a $p \in [0, 1]$ such that

$$u_1(\sigma_p|S) = 55 - 40p > B = u_1(M|S)$$

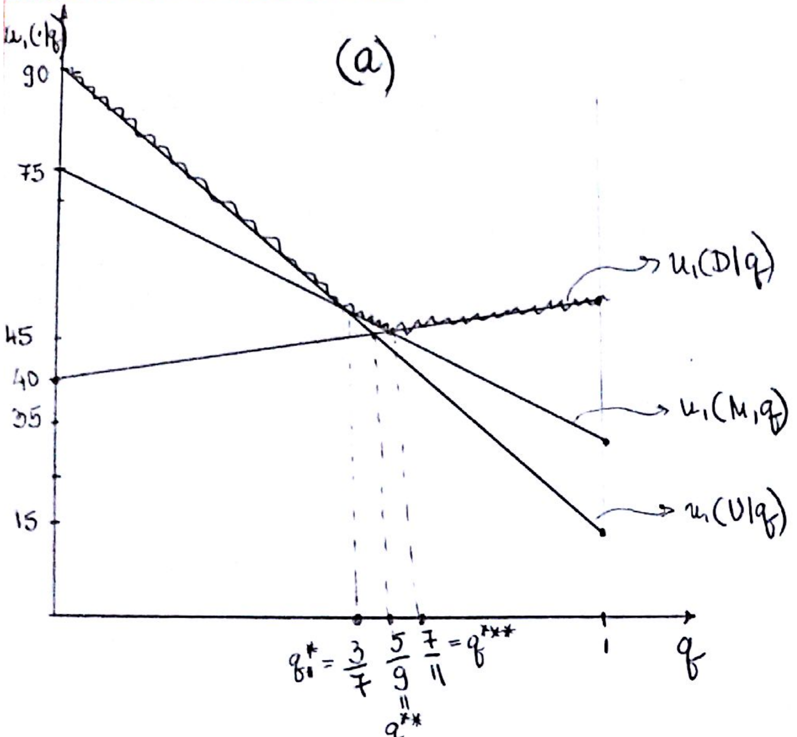
and

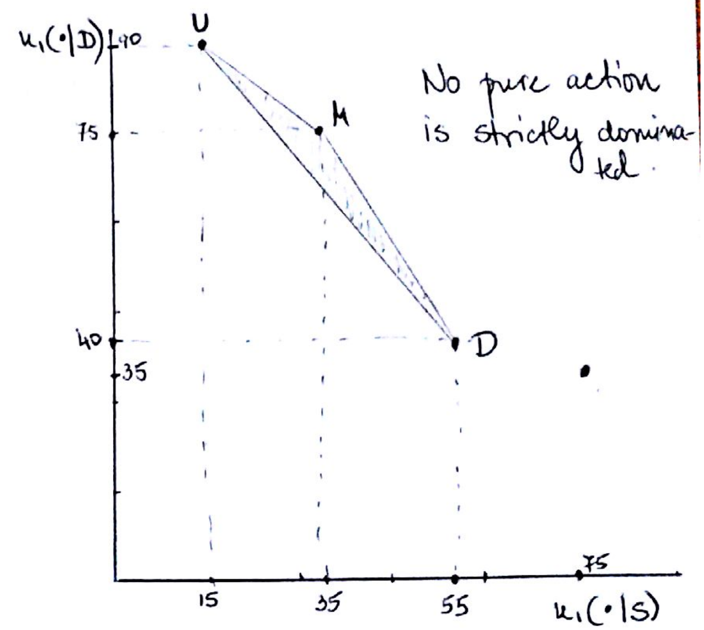
$$u_1(\sigma_p|D) = 40 + 50p > 75 = u_1(M|D).$$

The first inequality requires $p < \frac{55-B}{40}$, while the second inequality requires $p > \frac{7}{10}$. Therefore, there will be a p that satisfies both inequalities as long as

$$\frac{55 - B}{40} > \frac{7}{10} \Leftrightarrow 550 - 10B > 280 \Leftrightarrow B < 27.$$

That is, for any B that is less than 27, there is an action (possibly mixed) that strictly dominates M.





$\because u_1(U|q) = u_1(M|q)$

$90 - 75q = 75 - 40q$

$15 = 35q$

$q^* = \frac{15}{35} = \frac{3}{7}$

$\because u_1(U|q) = u_1(D|q)$

$90 - 75q = 40 + 15q$

$90q = 50$

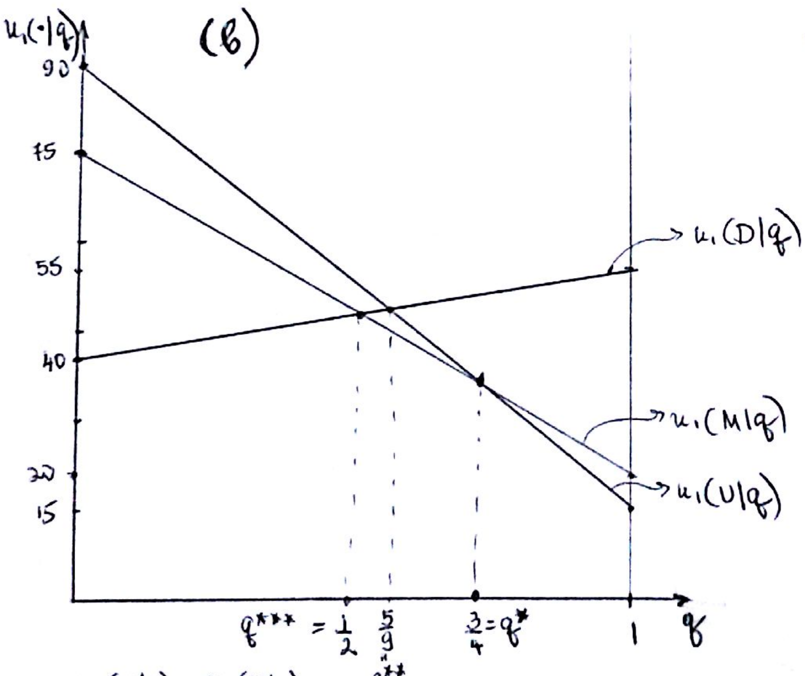
$q^{**} = \frac{5}{9}$

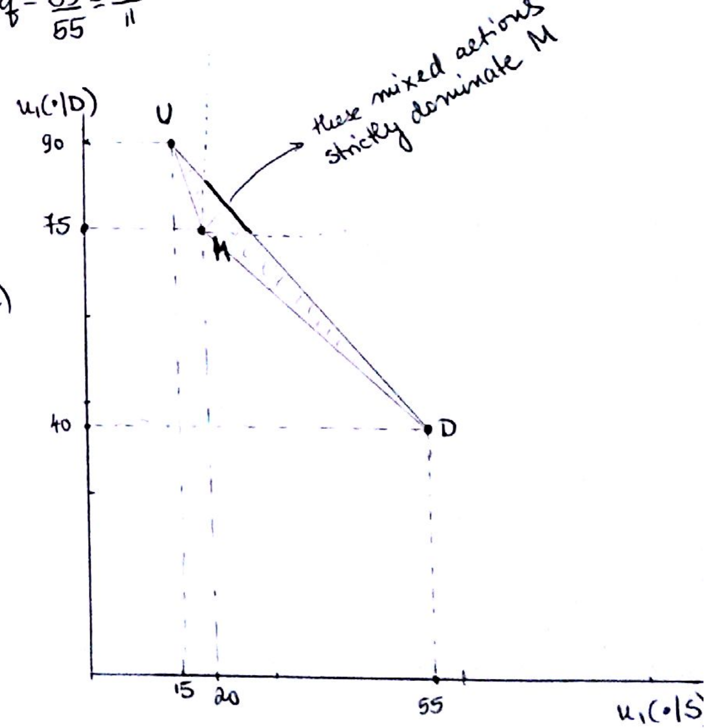
$\because u_1(M|q) = u_1(D|q)$

$75 - 40q = 40 + 15q$

$55q = 35$

$q^{***} = \frac{35}{55} = \frac{7}{11}$





q^{**}

$\because u_1(U|q) = u_1(M|q)$

$90 - 75q = 75 - 55q$

$15 = 20q$

$q^* = \frac{3}{4}$

$\because u_1(U|q) = u_1(D|q)$

$90 - 75q = 40 + 15q$

$90q = 50$

$q^{**} = \frac{5}{9}$

$\because u_1(M|q) = u_1(D|q)$

$75 - 55q = 40 + 15q$

$70q = 35$

$q^{***} = \frac{1}{2}$

Exercise 3: Solve the following game by iteratively deleting strictly dominated strategies:

		Player 2			
		a	b	c	d
Player 1	A	3,1	0,0	1,0	0,0
	B	1,1	1,0	1,1	1,2
	C	1,2	0,4	6,2	1,1
	D	0,4	1,0	1,1	2,3

There are no strategies that are strictly dominated by pure strategies for either player. Therefore, we need to check whether there are strategies dominated by mixed strategies. By Proposition 1.c.2 (p.7 of the lecture notes) we know that a strategy is strictly dominated by another (possibly mixed) strategy if and only if it is a NWBR. For Player 2, strategy c is a NWBR for any beliefs over the opponent's strategies. (Indeed, c is strictly dominated, for example, by the following mixed strategy that has only a, b and d in its support: $\sigma_2 = (\sigma_2(a), \sigma_2(b), \sigma_2(c), \sigma_2(d)) = (0.5, 0.2, 0, 0.3)$). Therefore, we can delete c to obtain:

		Player 2		
		a	b	d
Player 1	A	3,1	0,0	0,0
	B	1,1	1,0	1,2
	C	1,2	0,4	1,1
	D	0,4	1,0	2,3

Given this reduced payoff matrix, for Player 1, C is a NWBR (it is strictly dominated for example, by the following mixed strategy that has only A and D in its support: $\sigma_1 = (\sigma_1(A), \sigma_1(B), \sigma_1(C), \sigma_1(D)) = (0.4, 0, 0, 0.6)$). Therefore, we can delete C to obtain:

		Player 2		
		a	b	d
Player 1	A	3,1	0,0	0,0
	B	1,1	1,0	1,2
	D	0,4	1,0	2,3

Given this, for Player 2, b is strictly dominated by a, so we can delete b to obtain:

		Player 2	
		a	d
Player 1	A	3,1	0,0
	B	1,1	1,2
	D	0,4	2,3

Given this, for Player 1, B is a NWBR (it is strictly dominated, for example, by the following mixed strategy that has only A and D in its support:$\sigma_1 = (\sigma_1(A), \sigma_1(B), \sigma_1(D)) = (0.4, 0, 0.6)$). Therefore, we delete B to obtain

		Player 2	
		a	d
Player 1	A	3,1	0,0
	D	0,4	2,3

Given this, for Player 2, d is strictly dominated by a, so we delete it and obtain:

		Player 2
		a
Player 1	A	3,1
	D	0,4

Given this, D is strictly dominated by A so we can delete it to obtain

		Player 2
		a
Player 1	A	3,1

Hence, (A,a) is the unique strategy profile that survives the iterated deletion of strictly dominated strategies.

<u>Exercise 4:</u> Consider the following game:

		Player 2		
		L	C	R
Player 1	U	50,0	5,5	1,-1000
	D	50,50	5,0	0,-1000

Show that the set of strategies that survive the iterated deletion of *weakly* dominated strategies depends on the order of deletion.

If we start with Player 1: D is weakly dominated by U. Given that, both L and R are dominated (strictly in fact) by C. So we get U for Player 1and C for Player 2 as the strategies that survive (We get (U,C) as the unique solution in this case.)

If we start with Player 2: R is weakly dominated by both L and C. Given that, we cannot delete any further. So we get U and D for Player 1 and L and C for Player 2 as the strategies that survive. (We get (U,L), (U,C), (D,L), and (D,C) as solutions in this case.)

Notice, that when we started with Player 1, the iterated deletion of weakly dominated strategies eliminated one of the possible Nash equilibria of the game, namely (D,L). This would not have happened had we done iterated deletion of *strictly* dominated actions (which is basically what we do if we start with Player 2).

<u>Exercise 5:</u> Consider the following symmetric, two-player, simultaneous move game: each player i chooses an action from the set $A_1 = A_2 = \{100, 200, 300\}$. The payoffs are as follows:

$$u_i(a_i, a_{-i}) = \begin{cases} a_i + 200 & \text{if } a_i < a_{-i} \\ a_i & \text{if } a_i = a_{-i} \\ a_i - 200 & \text{if } a_i > a_{-i} \end{cases}$$

(a) Write down the normal form payoff matrix for this game.

		Player 2		
		100	200	300
	100	100,100	300,0	300,100
Player 1	200	0,300	200,200	400,100
	300	100,300	100,400	300,300

(b) Which actions are strictly dominated? Which actions are weakly dominated?

There are no strictly dominated actions. For both players, 300 is weakly dominated by 100.

(c) Find all of the pure-strategy Nash equilibria.

There is a unique pure-strategy equilibrium: (100,100)

(d) Find all of the Nash equilibria, including those in mixed strategies.

Consider mixing between all three actions. Denote $\sigma_1(100) = p_1$, $\sigma_1(200) = p_2$, $\sigma_1(300) = 1-p_1-p_2$, and $\sigma_2(100) = q_1$, $\sigma_2(200) = q_2$, $\sigma_2(300) = 1-q_1-q_2$.

The expected utilities of Player 1 from playing each pure action given the strategy σ_2 of his opponent are:
$u_1(100|\sigma_2) = 100q_1 + 300q_2 + 300(1-q_1-q_2) = 300 - 200q_1$
$u_1(200|\sigma_2) = 0q_1 + 200q_2 + 400(1-q_1-q_2) = 400 - 400q_1 - 200q_2$
$u_1(300|\sigma_2) = 100q_1 + 100q_2 + 300(1-q_1-q_2) = 300 - 200q_1 - 200q_2$

Suppose $q_2 > 0$. Then 300 is strictly dominated for Player 1. Therefore, he will set $1-p_1-p_2 = 0$ or $p_1+p_2 = 1$ and only mix between 100 and 200. But if Player 1 does not play 300, then for Player 2 200 becomes strictly dominated by 100, and hence he will set $q_2 = 0$. This is a contradiction. Hence, in any equilibrium, $q_2 = 0$ (and by symmetry $p_2 = 0$).

Given $q_2 = 0$, Player 1 is indifferent between 100 and 300 for any q_1. We just need to insure that 200 is never played with positive probability for this equilibrium to hold:
$300 - 200q_1 \geq 400 - 400q_1 \Leftrightarrow q_1 \geq \frac{1}{2}$.
Due to symmetry, the same holds from the perspective of Player 2. Hence, the set of mixed strategy NE is: $\big((p_1 \geq \frac{1}{2}, 0, 1-p_1); (q_1 \geq \frac{1}{2}, 0, 1-q_1)\big)$.
Notice that there are infinitely many mixed NE.

Exercise 6: Consider the following auction, known as a *second-price*, or *Vickrey*, auction. An object is auctioned off to N bidders. Bidder i's valuation of the object in monetary terms is v_i. The auction rules are that each player submit a bid (a non-negative number) in a sealed envelope. The envelopes are then opened, and the bidder who has submitted the highest bid gets the object but pays the auctioneer the amount of the second-highest bid. If more than one bidder submits the highest bid, each gets the object with equal probability. Show that submitting a bid of v_i with certainty is a weakly dominant strategy for bidder i. Also argue that this is bidder i's unique weakly dominant strategy.

Let us denote bidder i's bid by b_i.
Suppose $b_i > v_i$ and let us compare the payoffs of this strategy to the payoffs he would obtain if he were to bid v_i.

- If some other bidder j bids higher than b_i, then the two strategies give the same payoff of 0 for bidder i (he does not get the object).

- If the second highest bid is $b_j < v_i$, then the two strategies give the same payoff of $v_i - b_j$ for bidder i, since he wins the objects and pays b_j in both cases.

- If the second highest bid is higher than v_i, so that we have $b_i > b_j > v_i$, then, by bidding b_i bidder i has a payoff of $v_i - b_j < 0$ because he wins the object and pays a price higher than his valuation. If instead he were to bid v_i, he would not win the object and will have a payoff of zero.

Hence, bidding $b_i > v_i$ is weakly dominated by bidding v_i.

Suppose $b_i < v_i$ and let us compare the payoffs of this strategy to the payoffs he would obtain if he were to bid v_i.

- If the second highest bid is $b_j < b_i$, then the two strategies give the same payoff of $v_i - b_j$ for bidder i, since he wins the objects and pays b_j in both cases.

- If some other bidder j bids higher than v_i, then the two strategies give the same payoff of 0 for bidder i as he does not get the object regardless of whether he bid b_i or v_i.

- If the highest bid of the other bidders, denoted as $\bar{b}$, is higher than b_i but lower than v_i, that is $b_i < \bar{b} < v_i$, then, by bidding b_i bidder i has a payoff of 0. If instead he were to bid v_i, he would win the object, pay $\bar{b}$, and obtain a payoff of $v_i - \bar{b} > 0$.

Hence, bidding $b_i < v_i$ is weakly dominated by bidding v_i.

This argument implies that bidding v_i is the unique weakly dominant strategy.

Exercise 7: Consumers are uniformly distributed along a boardwalk that is 1 mile long. Ice cream prices are regulated so consumers go to the nearest vendor because they dislike walking. Assume that at the regulated price all consumers will purchase an ice cream even if they have to walk a full mile. If more than one vendor is at the same location, they split the business evenly.

(a) Consider a game in which two ice cream vendors pick their locations simultaneously. Show that there exists a *unique* pure strategy Nash equilibrium and that it involves both vendors locating at the midpoint of the boardwalk.

Let x_1 be the location of Vendor 1 and x_2 be the location of Vendor 2 measured as the distance from the same end of the boardwalk. Thus, the strategy for Player i can be represented as $x_i \in [0, 1]$.

Since the price of ice cream is regulated, the profit for each vendor can be identified by the proportion of consumers he gets.

– Suppose $x_1 < x_2$. Then all consumers located to the left of $\frac{x_1+x_2}{2}$ will purchase from Vendor 1, while all consumers located to the right of $\frac{x_1+x_2}{2}$ will buy from Vendor 2. The corresponding fractions of consumers (given the uniform distribution) are:

$$u_1(x_1, x_2) = \frac{x_1 + x_2}{2}$$

$$u_2(x_1, x_2) = 1 - \frac{x_1 + x_2}{2}$$

- Suppose $x_1 > x_2$. Then all consumers located to the right of $\frac{x_1+x_2}{2}$ will purchase from Vendor 1, while all consumers located to the left of $\frac{x_1+x_2}{2}$ will buy from Vendor 2. The corresponding fractions of consumers (given the uniform distribution) are:

$$u_1(x_1, x_2) = 1 - \frac{x_1 + x_2}{2}$$

$$u_2(x_1, x_2) = \frac{x_1 + x_2}{2}$$

- Suppose $x_1 = x_2$. Then the vendors split the business so that

$$u_1(x_1, x_2) = u_2(x_1, x_2) = \frac{1}{2}$$

Hence, we can summarize this in the following payoff function:

$$u_i(x_i, x_{-i}) = \begin{cases} \frac{x_i+x_{-i}}{2} & \text{if } x_i < x_{-i} \\ \frac{1}{2} & \text{if } x_i = x_{-i} \\ 1 - \frac{x_i+x_{-i}}{2} & \text{if } x_i > x_{-i} \end{cases}$$

It is straightforward to check that $x_1 = x_2 = \frac{1}{2}$ in indeed an equilibrium: no vendor can do better by deviating.

To show uniqueness we will consider all the different possibilities:

- suppose $x_1 = x_2 < \frac{1}{2}$ is an equilibrium. Then either vendor can do better by moving by $\varepsilon > 0$ to the right, since it will sell almost $1 - x_1 > \frac{1}{2}$ units rather than $\frac{1}{2}$ units. Hence, there is a profitable deviation $\rightarrow$ a contradiction.
- suppose $x_1 = x_2 > \frac{1}{2}$ is an equilibrium. Then either vendor can do better by moving by $\varepsilon > 0$ to the left, since it will sell almost $x_1 > \frac{1}{2}$ units rather than $\frac{1}{2}$ units. Hence, there is a profitable deviation $\rightarrow$ a contradiction.
- suppose $x_1 < x_2$ is an equilibrium. Then Vendor 1 can do better by moving to $x_2 - \varepsilon$ with $\varepsilon > 0$. Hence, there is a profitable deviation $\rightarrow$ a contradiction.
- suppose $x_1 > x_2$ is an equilibrium. Then Vendor 2 can do better by moving to $x_1 + \varepsilon$ with $\varepsilon > 0$. Hence, there is a profitable deviation $\rightarrow$ a contradiction.

Therefore, there is no other equilibrium.

(b) Show that with three vendors, no pure strategy Nash equilibrium exists.

Suppose that an equilibrium (x_1^*, x_2^*, x_3^*) exists. We will consider all different possibilities.

- suppose $x_1^* = x_2^* = x_3^*$. Then each vendor will have a payoff of $\frac{1}{3}$. But any vendor can increase its sales by moving to the right (if $x_1^* = x_2^* = x_3^* < \frac{1}{2}$) or to the left (if $x_1^* = x_2^* = x_3^* \geq \frac{1}{2}$). Hence, there is a profitable deviation $\rightarrow$ a contradiction.
- suppose two vendors locate at the same point, let's say $x_1^* = x_2^*$. If $x_1^* = x_2^* < x_3^*$, then Vendor 3 can do better by moving to $x_1^* + \varepsilon$. If $x_1^* = x_2^* > x_3^*$, then Vendor 3 can do better by moving to $x_1^* - \varepsilon$. Hence, there is a profitable deviation $\rightarrow$ a contradiction.

– suppose that all three vendors locate at different points. But then, the vendor that is located the farthest on the right will be able to increase his sales by moving to the left by $\varepsilon > 0$. Hence, there is a profitable deviation $\rightarrow$ a contradiction.

Thus, there exists no pure strategy NE in this game.