Solutions to Problem Set 1

Microeconomics I Part B

Exercise 1: Suppose the set of outcomes is given by $C = \{10, 20, 30\}$, and consider the lottery given by $L = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

(a) Find the expectation and variance of L.

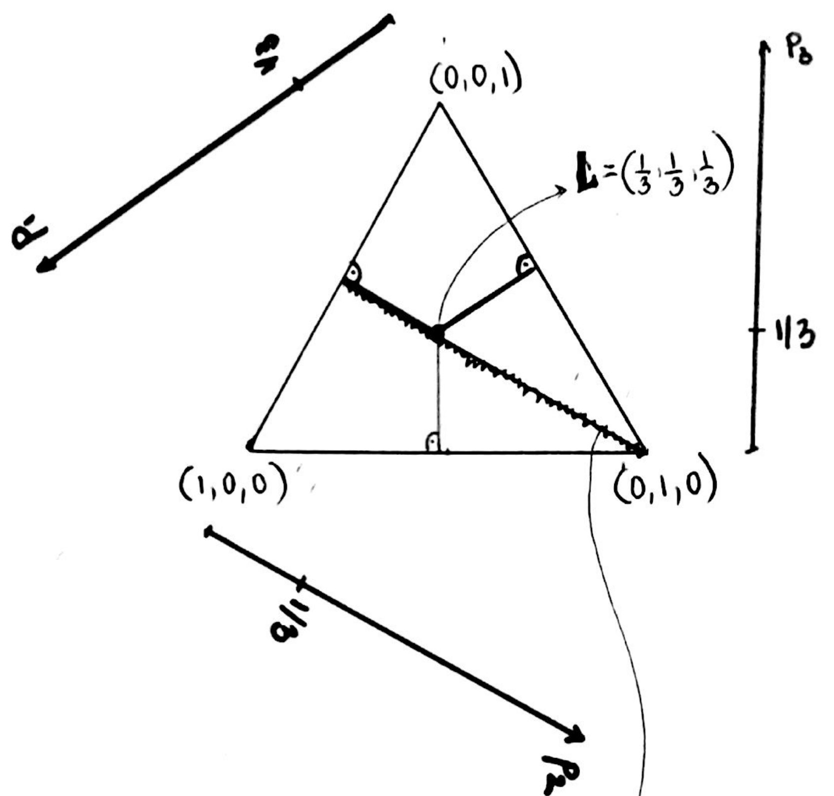
$$E(L) = \frac{1}{3} \cdot 10 + \frac{1}{3} \cdot 20 + \frac{1}{3} \cdot 30 = 20$$

$$Var(L) = \frac{1}{3} \cdot (10 - 20)^2 + \frac{1}{3} \cdot (20 - 20)^2 + \frac{1}{3} \cdot (30 - 20)^2 = 66.67$$

(b) Draw L in the 2-dimensional simplex.
(see attached drawing)

(c) Draw the set of all lotteries on C that have the same expectation as L.
(see attached drawing)

Exercise 1: (b) and (c)



(c) $p_1 \cdot 10 + p_2 \cdot 20 + p_3 \cdot 30 = 20$

$\Leftrightarrow p_1 \cdot 10 + (1 - p_1 - p_3) \cdot 20 + p_3 \cdot 30 = 20$

$\Leftrightarrow 10p_3 - 10p_1 = 0 \Leftrightarrow p_1 = p_3$

$\{(p_1, p_2, p_3) \mid 10p_1 + 20p_2 + 30p_3 = 20, p_1 + p_2 + p_3 = 1\}$

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Exercise 2: An individual has Bernoulli utility function $u(\cdot)$ and initial wealth w. Let lottery L offer a payoff of G with probability p and a payoff of B with probability $1-p$.

(a) If the individual owns the lottery, set up the equation which determines the minimum price he would sell it for?
If the individual owns the lottery, his random wealth will be $w+G$ with probability p or $w+B$ with probability $1-p$. Thus, the minimal selling price R_s is determined by the following equation:

$$pu(w+G)+(1-p)u(w+B)=u(w+R_s).$$

(b) If he does not own it, set up the equation which determines the maximum price he would be willing to pay for it?
If the individual buys the lottery at price R_b, then his random wealth will be $w+G-R_b$ with probability p and $w+B-R_b$ with probability $1-p$. The maximum buying price is determined by the following equation:

$$u(w)=pu(w+G-R_b)+(1-p)u(w+B-R_b).$$

In general, the two prices are not equal.

Exercise 3: Show that if the preferences $\succeq$ over $\mathscr{L}$ is represented by a utility function $U(\cdot)$ that has the expected utility form, then $\succeq$ satisfies the independence axiom.

Assume that the preference relation $\succeq$ is represented by a v.N-M utility function. Hence, for every $L\in\mathscr{L}$ we have $U(L)=\sum_{n=1}^{N}p_nu_n$. Let $L=(p_1,\dots,p_N)\in\mathscr{L}$, $L'=(p'_1,\dots,p'_N)\in\mathscr{L}$, $L''=(p''_1,\dots,p''_N)\in\mathscr{L}$, and $\alpha\in(0,1)$. Then $L\succeq L'$ if and only if $\sum_{n=1}^{N}p_nu_n\geq\sum_{n=1}^{N}p'_nu_n$. Multiplying both sides by α and adding $(1-\alpha)\left(\sum_{n=1}^{N}p''_nu_n\right)$ to both sides, this inequality is equivalent to

$$\alpha\left(\sum_{n=1}^{N}p_nu_n\right)+(1-\alpha)\left(\sum_{n=1}^{N}p''_nu_n\right)\geq\alpha\left(\sum_{n=1}^{N}p'_nu_n\right)+(1-\alpha)\left(\sum_{n=1}^{N}p''_nu_n\right).$$

The last inequality holds if and only if $\alpha L+(1-\alpha)L''\succeq\alpha L'+(1-\alpha)L''$. We thus obtain that $L\succeq L'$ if and only if $\alpha L+(1-\alpha)L''\succeq\alpha L'+(1-\alpha)L''$. Therefore, the independence axiom holds.

Exercise 4: Suppose that a safety agency is thinking of establishing a criterion under which an area prone to flooding should be evacuated. The probability of flooding is 1%. There are four possible outcomes:

(A) No evacuation is necessary, and none is performed.

(B) An evacuation is performed that is unnessesary.

(C) An evacuation is performed that is necessary.

(D) No evacuation is performed, and a flood causes a disaster.

Suppose the agency is indifferent between the sure outcome B and the lottery of A with probability p and D with probability $1-p$, and between the sure outcome C and the lottery of B with probability q and D with probability $1-q$. Suppose also that it prefers A to D and that $p \in (0,1)$ and $q \in (0,1)$. Assume that the conditions of the expected utility theorem are satisfied.

(a) Construct a utility function of the expected utility form for the agency.

We are given that $A \succ D$, $B \sim (A, D; p, (1-p))$, $C \sim (B, D; q, (1-q))$ where $p \in (0,1)$, $q \in (0,1)$.
Let $u(A) = 1$ and $u(D) = 0$. Then,

$$u(B) = pu(A) + (1-p)u(D) = p$$

and

$$u(C) = qu(B) + (1-q)uD = qp.$$

(b) Consider two different policy criteria:

Criterion 1(C^1): This criterion will result in an evacuation in 90% of the cases in which flooding will occur and an unnecessary evacuation in 10% of the cases in which no flooding occurs.

Criterion 2 (C^2): This criterion is more conservative. It will result in an evacuation in 95% of the cases in which flooding will occur and an unnecessary evacuation in 15% of the cases in which no flooding occurs.

First, derive the probability distribution over the four outcomes under these two criteria. Then, by using the utility function in (a), decide which criterion the agency would prefer.

Criterion 1(C^1): $p_A^1 = 0.99 \cdot 0.9 = 0.891$; $p_B^1 = 0.99 \cdot 0.1 = 0.099$;
$p_C^1 = 0.01 \cdot 0.9 = 0.009$; $p_D^1 = 0.01 \cdot 0.1 = 0.001$. Hence, we have
$C^1 = (p_A^1, p_B^1, p_C^1, p_D^1) = (0.891, 0.099, 0.009, 0.001)$.

Criterion 2(C^2): $p_A^2 = 0.99 \cdot 0.85 = 0.8415$; $p_B^2 = 0.99 \cdot 0.15 = 0.1485$;
$p_C^2 = 0.01 \cdot 0.95 = 0.0095$; $p_D^2 = 0.01 \cdot 0.05 = 0.0005$. Hence, we have
$C^2 = (p_A^2, p_B^2, p_C^2, p_D^2) = (0.8415, 0.1485, 0.0095, 0.0005)$.

$u(C^1) = 0.891u_A + 0.099u_B + 0.009u_C + 0.001u_D = 0.891 + 0.099p + 0.009pq$
$u(C^2) = 0.8415u_A + 0.1485u_B + 0.0095u_C + 0.0005u_D = 0.8415 + 0.1485p + 0.0095pq$
$u(C^1) - u(C^2) = 0.0495 - 0.0495p - 0.0005pq = 0.0495 - 0.0005p(99+q)$
$u(C^1) \geq u(C^2) \Leftrightarrow 0.0495 \geq 0.0005p(99+q) \Leftrightarrow 99 \geq p(99+q) \Leftrightarrow \frac{99}{99+q} \geq p$
Hence, the agency weakly prefers Criterion 1 if $\frac{99}{99+q} \geq p$, and prefers Criterion 2 otherwise.

<u>Exercise 5:</u> Suppose that an individual has a Bernoulli utility function $u(x) = \sqrt{x}$.

(a) Calculate the certainty equivalent and the risk premium for a gamble $(16, 4; \frac{1}{2}, \frac{1}{2})$.

$$u(c) = \frac{1}{2} \cdot u(16) + \frac{1}{2} \cdot u(4) \Leftrightarrow \sqrt{c} = \frac{1}{2} \cdot \sqrt{16} + \frac{1}{2} \cdot \sqrt{4} \Leftrightarrow \sqrt{c} = 3 \Leftrightarrow c = 9$$

$$rp = \frac{1}{2} \cdot 16 + \frac{1}{2} \cdot 4 - c = 8 + 2 - 9 = 1$$

(b) Calculate the certainty equivalent and the risk premium for a gamble $(36, 16; \frac{1}{2}, \frac{1}{2})$.

$$u(c) = \frac{1}{2} \cdot u(36) + \frac{1}{2} \cdot u(16) \Leftrightarrow \sqrt{c} = \frac{1}{2} \cdot \sqrt{36} + \frac{1}{2} \cdot \sqrt{16} \Leftrightarrow \sqrt{c} = 5 \Leftrightarrow c = 25$$

$$rp = \frac{1}{2} \cdot 36 + \frac{1}{2} \cdot 16 - c = 18 + 8 - 25 = 1$$

<u>Exercise 6:</u> Wendy the Witch is a strictly risk averse expected utility maximizer. Her wealth is w, but with probability π children will eat her gingerbread house, causing her a loss of D. She can buy an insurance policy that pays \$1 per unit of insurance in the event of loss at a price of q per unit, so the total cost if she buys α units is αq. She also can invest in a home security system. Each dollar she spends on the home security system reduces the probability of the loss by c, so if she spends s on the security, the probability that her house will be eaten is $\pi - cs$. The maximum amount she can spend on security is π/c. You may assume that the wealth constraint does not bind, i.e. that Wendy can purchase full insurance and the maximal level of investment in the home security system with her wealth $(w > Dq + \pi/c)$.

(a) Set up Wendy's expected utility maximization problem. Derive the Kuhn-Tucker (K-T) conditions that characterize the solutions $\alpha^*(w, \pi, D, q, c)$ and $s^*(w, \pi, D, q, c)$.

$$\max_{\alpha, s} \quad (1 - \pi + cs)u(w - \alpha q - s) + (\pi - cs)u(w - \alpha q - s - D + \alpha)$$

subject to

$$\alpha \geq 0; \quad \frac{\pi}{c} \geq s \geq 0; \quad w - \alpha q - s \geq 0$$

Therefore, the Lagrangian is

$$\mathbb{L} = (1-\pi+cs)u(w-\alpha q-s)+(\pi-cs)u(w-\alpha q-s-D+\alpha)+\lambda_1\alpha+\lambda_2(\frac{\pi}{c}-s)+\lambda_3 s+\lambda_4(w-\alpha q-s)$$

FOC:

$$\frac{\partial \mathbb{L}}{\partial \alpha} = -q(1-\pi+cs)u'(w-\alpha q-s)+(1-q)(\pi-cs)u'(w-\alpha q-s-D+\alpha)+\lambda_1-\lambda_4 q = 0 \quad (1)$$

$$\frac{\partial \mathbb{L}}{\partial s} = cu(w - \alpha q - s) - (1 - \pi + cs)u'(w - \alpha q - s) - cu(w - \alpha q - s - D + \alpha) - (\pi - cs)u'(w - \alpha q - s - D + \alpha) - \lambda_2 + \lambda_3 - \lambda_4 = 0 \quad (2)$$

$$\frac{\partial \mathbb{L}}{\partial \lambda_1} = \alpha \geq 0 \qquad \lambda_1 \geq 0 \qquad \lambda_1 \alpha = 0 \quad (3)$$

$$\frac{\partial \mathbb{L}}{\partial \lambda_2} = \frac{\pi}{c} - s \geq 0 \qquad \lambda_2 \geq 0 \qquad \lambda_2(\frac{\pi}{c} - s) = 0 \quad (4)$$

$$\frac{\partial \mathbb{L}}{\partial \lambda_3} = s \geq 0 \qquad \lambda_3 \geq 0 \qquad \lambda_3 s = 0 \quad (5)$$

$$\frac{\partial \mathbb{L}}{\partial \lambda_4} = w - \alpha q - s \geq 0 \qquad \lambda_4 \geq 0 \qquad \lambda_4(w - \alpha q - s) = 0 \quad (6)$$

(b) Use the K-T conditions to show that if the maximum security constraint binds ($s^* = \pi/c$), then the optimal amount of insurance α^* is zero. Explain intuitively.

If $s^* = \frac{\pi}{c}$, then the FOC w.r.t. α (equation (1)) becomes:

$$\frac{\partial \mathbb{L}}{\partial \alpha} = -qu'(w - \alpha q - \frac{\pi}{c}) + \lambda_1 - \lambda_4 q = 0$$

which gives

$$\lambda_1 = qu'(w - \alpha q - \frac{\pi}{c}) + \lambda_4 q > 0.$$

Since by (3) we must have $\lambda_1 \alpha = 0$ and we just showed that $\lambda_1 > 0$, it must be that $\alpha^* = 0$.

Intuitively, this makes sense, since if $s^* = \frac{\pi}{c}$, the probability of loss is zero and there is no point in spending money on insuring against the loss.

(c) Use the K-T conditions to show that if Wendy buys full insurance ($\alpha^* = D$), then the optimal amount of security s^* is zero. Explain intuitively.

If $\alpha^* = D$, then the FOC w.r.t. s (equation (2)) becomes:

$$\frac{\partial \mathbb{L}}{\partial s} = -u'(w - Dq - s) - \lambda_2 + \lambda_3 - \lambda_4 = 0$$

which gives

$$\lambda_3 - \lambda_2 = u'(w - Dq - s) + \lambda_4 > 0.$$

This means $\lambda_3 > \lambda_2 \geq 0$. Since by (5) we must have $\lambda_3 s = 0$, it must be that $s^* = 0$.

Intuitively, this makes sense, since for $\alpha^* = D$ the loss is fully insured and there is no point in spending money on reducing the probability of loss.

(d) Use the K-T conditions to show that if $q > \pi$, then Wendy will not buy full insurance. (That is, $q > \pi \Rightarrow \alpha^* \neq D$.)

We will prove this by contradiction. Suppose $q > \pi$ and $\alpha^* = D$. From part (c) we know that then we must have $s^* = 0$. Since $\alpha^* = D > 0$, we must have $\lambda_1 = 0$. The FOC w.r.t. α (equation (1)) then becomes

$$\frac{\partial \mathbb{L}}{\partial \alpha} = -q(1-\pi)u'(w - Dq) + (1-q)\pi u'(w - Dq) - \lambda_4 q = 0$$

which gives

$$(\pi - q)u'(w - Dq) = \lambda_4 q. \tag{7}$$

Since the wealth constraint does not bind, we have that for $\alpha^* = D$ and $s^* = 0$, $w - Dq > 0$. By (6), we must have $\lambda_4(w - Dq) = 0$, and hence $\lambda_4 = 0$. But then, for (7) to hold, we must have $\pi = q$ (since $u' > 0$). This is a contradiction to $q > \pi$.