Section B

Question 1. Sonja is an expected utility maximizer whose Bernoulli utility function is given by $u(x) = x^2$. She has wealth $w > 0$, but with probability $\pi > 0$ she suffers a loss of $D > 0$ pounds. She can buy insurance. One unit of insurance pays off 1 pound in case of loss and costs p pounds ($p > 0$), so if she buys a units of insurance, she pays ap pounds. She cannot buy negative insurance, nor can she spend more than w on insurance. She can afford full insurance if she wanted to at any $p \in [0, 1]$ and she will still have some money left over, as she has had a very successful career.

(a) Determine the risk attitude of Sonja (i.e., is she strictly/weakly risk-averse, risk-loving or risk-neutral)?
$u''(x) = 2 > 0$, so $u(x)$ is strictly convex, which means that Sonja is strictly risk-loving.

(b) Set up Sonja's expected utility maximization problem. Write out the associated Lagrangian, and first-order (Kuhn-Tucker) conditions.

$$\max_a \quad \pi(w - D + a - ap)^2 + (1 - \pi)(w - ap)^2 \tag{1}$$

subject to

$$0 \leq ap \leq w$$

Therefore, the Lagrangian is

$$\mathbb{L}(a, \lambda_1, \lambda_2) = \pi(w - D + a - ap)^2 + (1 - \pi)(w - ap)^2 + \lambda_1 ap + \lambda_2(w - ap) \tag{2}$$

FOC:

$$\frac{\partial \mathbb{L}}{\partial a} = 2(1 - p)\pi(w - D + a - ap) - 2p(1 - \pi)(w - ap) + \lambda_1 p - \lambda_2 p = 0 \tag{3}$$

$$\frac{\partial \mathbb{L}}{\partial \lambda_1} = ap \geq 0 \qquad \lambda_1 \geq 0 \qquad \lambda_1 ap = 0 \tag{4}$$

$$\frac{\partial \mathbb{L}}{\partial \lambda_2} = w - ap \geq 0 \qquad \lambda_2 \geq 0 \qquad \lambda_2(w - ap) = 0 \tag{5}$$

(c) Show that if $p = \pi$, then there is a solution to the first-order conditions where the amount of insurance that Sonja purchases is equal to D (that is, full insurance).

Suppose that $a = D$. Then $\lambda_1 = \lambda_2 = 0$. Substituting this (and also $p = \pi$) into (3) we get:

$$2(1-p)p(w - D + a - ap) = 2p(1-p)(w - ap)$$

which in turn implies:

$$w - D + a - ap = w - ap \Leftrightarrow a = D$$

Hence, $a = D$ satisfies all of the Kuhn-Tucker conditions.

(d) Show, however, that if $p = \pi$, then full insurance does NOT maximize Sonja's expected utility. (HINT: Use second derivatives to show that at $a = D$, the value of the Lagrangian increases when a increases.)

$$\frac{\partial^2 \mathbb{L}}{\partial a^2} = 2(1-p)^2\pi + 2p^2(1-\pi). \tag{6}$$

Evaluated at $a = D$ and $p = \pi$ this derivative is

$$\frac{\partial^2 \mathbb{L}}{\partial a^2} = 2(1-p)p > 0. \tag{7}$$

Hence, increasing a increases expected utility and so $a = D$ does not maximize expected utility.

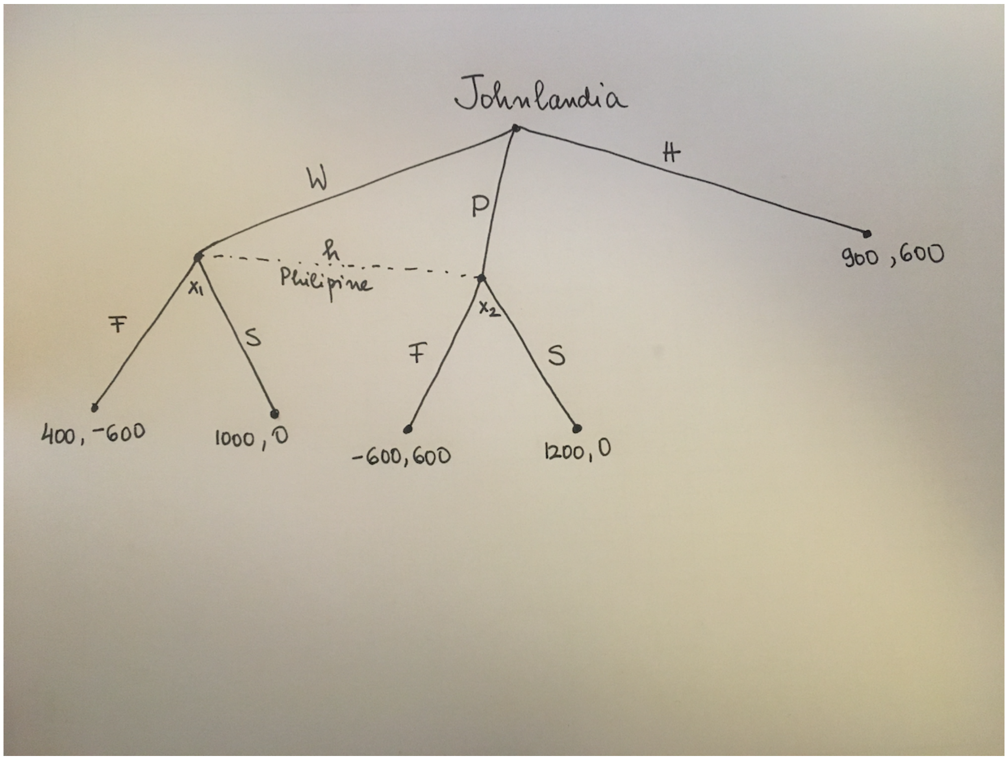
(e) Explain in words why Sonja prefers being over-insured ($a > D$) to having full insurance ($a = D$) when $p = \pi$.

When the policy is actuarially fair, Sonja's expected wealth is independent of a (indeed it is equal to $\pi(w - D + a - a\pi) + (1-\pi)(w - a\pi) = w - D\pi$). So by raising a above D she can increase risk (which she loves) without sacrificing expected wealth. Basically, she is facing the lottery of receiving $w - D + (1-\pi)a$ with probability π or $w - a\pi$ with probability $1 - \pi$, and by increasing a beyond D she makes the first payoff larger and the second payoff smaller, without affecting the expected payoff of the lottery.

Question 2. The country of Johnlandia is deciding how and whether to attack the country of Philipine. Johnlandia has three choices: it can 1) send a well-equipped invasion force (denoted by W), 2) send a poorly-equipped invasion force (denoted by P), or 3) stay at home (denoted by H). A well-equipped invasion force is more expensive than a poorly equipped one, but it is also more likely to win in battle. If Johnlandia chooses to invade, then Philipine has two choices: it can either fight (denoted by F) or surrender (denoted by S). Crucially, when making its decision, Philipine cannot tell whether Johnlandia sent a well-equipped or a poorly-equipped invasion force. That fact is common knowledge as are the following payoffs:

> If Johnlandia stays home, its payoff is 900 and Philipine's is 600.
>
> If Johnlandia sends a well-equipped force and Philipine fights, payoffs are 400 and –600 respectively.
>
> If Johnlandia sends a poorly-equipped force and Philipine fights, payoffs are –600 and 600 respectively.
>
> If Johnlandia sends a well-equipped force and Philipine surrenders, payoffs are 1000 and 0 respectively.
>
> If Johnlandia sends a poorly-equipped force and Philipine surrenders, payoffs are 1200 and 0 respectively.

(a) Draw the extensive form of this game. What are the sets of pure strategies for each of the two countries? Can backwards induction be used to solve for the subgame perfect equilibria of this game? Explain briefly.



The sets of pure strategies for each of the two countries: $S_J = \{H, W, P\}$, $S_P = \{F, S\}$ No we cannot use backwards induction here, as this is a game of imperfect information (there are non-trivial information sets).

(b) Find all pure strategy subgame perfect equilibria. Explain briefly.

Since there are no proper subgames, all pure strategy equilibria are also subgame perfect. The corresponding normal form of the game:

		Philipine F	Philipine S
Johnlandia	H	900,600	900,600
	W	400,–600	1000,0
	P	–600,600	1200,0

Pure strategy NE: (H, F)

Hence, the only pure strategy SPE is (H, F).

(c) Find all pure strategy perfect Bayesian equilibria.

Since there is a unique pure strategy SPE, that is the only candidate for the strategy part of the pure strategy PBE. We need to make sure that Philipine is being sequentially rational at h. Since according to this strategy, information set h is never reached, we can choose the beliefs however we like. Denote $\mu_P(x_1|h) = \mu$. The expected payoff from playing F at h is:

$$EP_P(F|h) = \mu \cdot (-600) + (1-\mu) \cdot 600 = 600 - 1200\mu$$

while from playing S at h is:

$$EP_P(S|h) = \mu \cdot 0 + (1-\mu) \cdot 0 = 0.$$

Hence, for any $\mu \leq \frac{1}{2}$, choosing F at h would be sequentially rational for Philipine. It remains to show that Johnlandia is being sequentially rational given the strategy of the opponent, that is given Philipine choosing F at h: H gives a payoff of 900, which is higher than the payoff from choosing W (400) and the payoff from choosing P (–600).

Hence, $((H, F), \mu_P(x_1) = \mu \leq 0.5, \mu_P(x_2) = 1-\mu)$ is the set of pure strategy PBE.

(d) Find a perfect Bayesian equilibrium in which Johnlandia stays at home with probability one and Philipine randomizes non-trivially (i.e. puts strictly positive probabilities on more than one action).

For Philipine to randomize, they need to be indifferent between F and S, that is:

$$EP_P(F|h) = EP_P(S|h)$$

which holds for $\mu = 0.5$. Under this belief, Philipine will be willing to mix. Denote my $\sigma_P(F) = p$, and hence $\sigma_P(S) = 1-p$. To make H sequentially rational for Johnlandia the following two conditions need to hold:

$$EP_J(W) = p \cdot 400 + (1-p) \cdot 1000 = 1000 - 600p \leq 900 = EP_J(H)$$

$$EP_J(P) = p \cdot (-600) + (1-p) \cdot 1200 = 1200 - 1800p \leq 900 = EP_J(H)$$

They both require $p \geq \frac{1}{6}$. Therefore, any of the equilibria in the following set is a possible answer:
$\big((H, \sigma_P(F) \in [\frac{1}{6}, 1), \sigma_P(S) = 1 - \sigma_P(F)), \mu_P(x_1) = \mu_P(x_2) = 0.5\big)$

(e) Find a perfect Bayesian equilibrium in which both Johnlandia and Philipine randomize with positive probability on all the actions in their respective action sets (that is, the supports of their mixed strategies are equal to their respective action sets).
Denote $\sigma_P(F) = p$, and hence $\sigma_P(S) = 1-p$. For Johnlandia to randomize btween all actions, they need to be indifferent:

$$EP_J(W) = EP_J(H) = EP_J(P),$$

which holds for $p = \frac{1}{6}$.

The indifference condition for Philipine requires that $\mu_P(x_1|h) = \frac{1}{2}$. Therefore, we need to have $\sigma_J(W) = \sigma_J(P)$ to ensure that these beliefs are consistent with Bayes rule as now h is reached with positive probability. Hence, any equilibrium in the following set is a possible answer:
$\big((\sigma_J(H) = q, \sigma_J(W) = \sigma_J(P) = \frac{1-q}{2}, \sigma_P(F) = \frac{1}{6}, \sigma_P(S) = \frac{5}{6}), \mu_P(x_1) = \mu_P(x_2) = 0.5\big)$