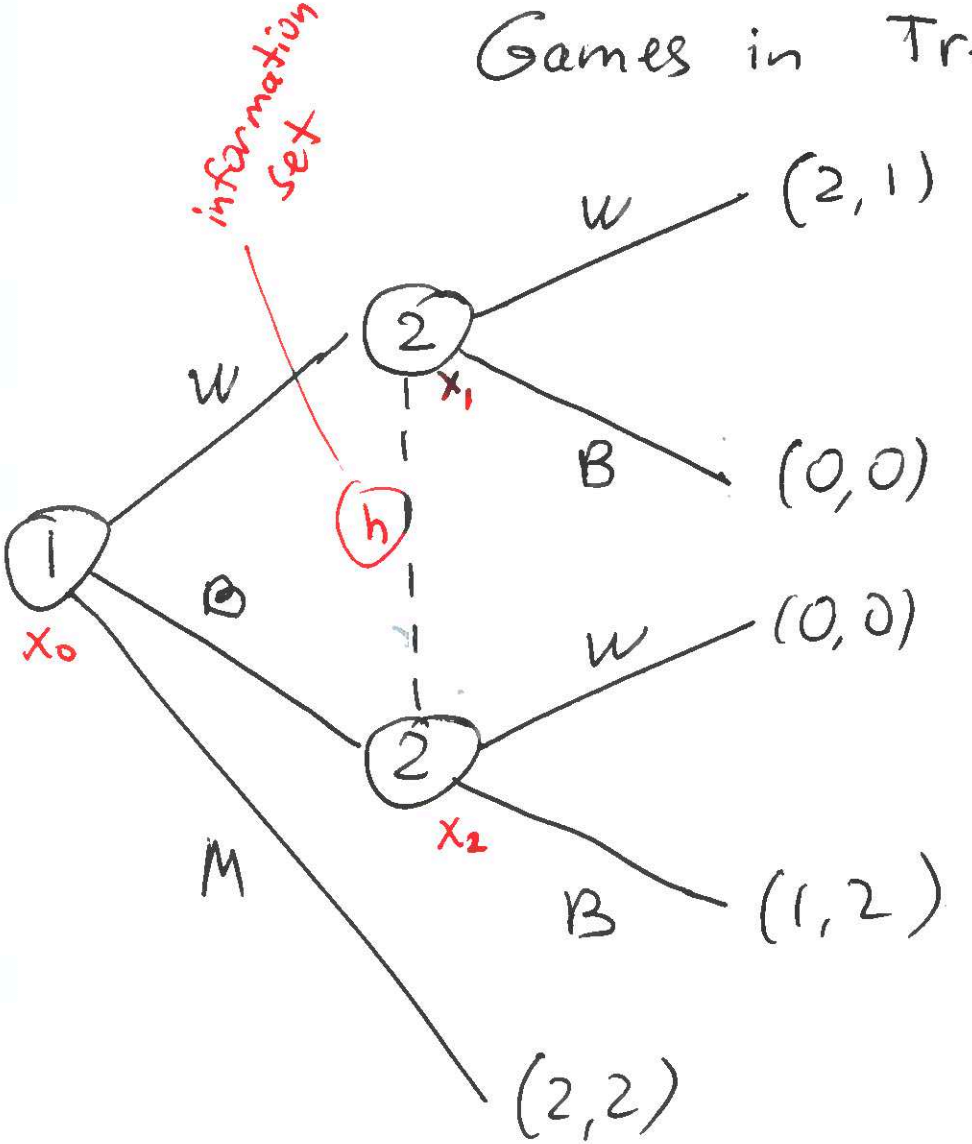
Extensive-form Games

Games in Tree Form

→ simultaneous
→ sequential



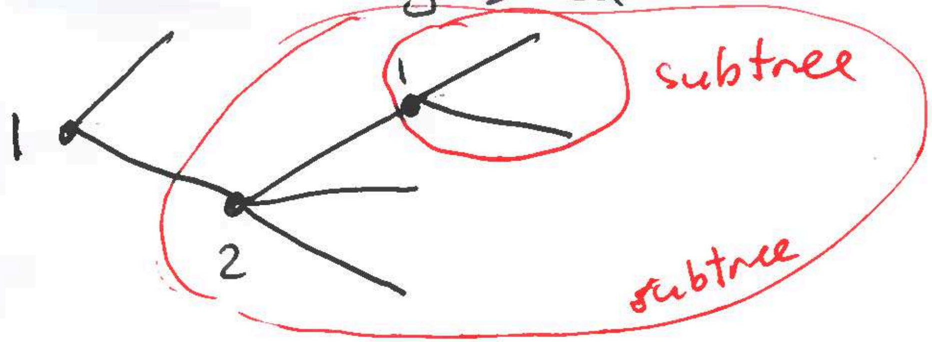
Ingredients of an Extensive-Form Game

1. Players $0, 1, 2, \ldots, N$ Player 0 is Nature

2. Nodes
   - Decision nodes ($x \in X$, initial node x_0)
   - Terminal nodes ($z \in Z$)

3. The tree:
   → each node has one predecessor (except for x_0)
   → each decision node has two or more successors
   → each terminal node has no successors

<u>Subtree</u> is a part of the tree that comprises a decision node and all its successors and all successors of successors etc.



The whole tree is a subtree of itself.

4. Decision mapping $i : X \to \{0, 1, 2, \ldots, N\}$
that specifies which player gets to move in each decision node

$i(x)$ = player who moves in node x

5. Actions: $A(x)$ is the set of actions available to player $i(x)$ at decision node $x \in X$.

6. Information sets: What a player knows about moves made earlier in the game.

<u>Info sets:</u> Let H be a partition of the decision nodes

↳ A player cannot distinguish between the decision nodes within the information set she is at, but can distinguish between info. sets.

$h(x) =$ info. set that contains node x.

Constraints

(a) In an info. set, the same player moves in each node of that set.

(b) In an info. set, the same actions are available in each node of that set.

if $x' \in h(x)$, then $i(x') = i(x)$

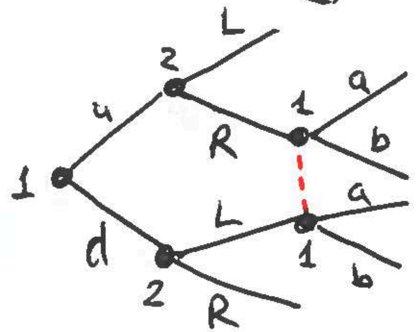
if $x' \in h(x)$, then $A(x') = A(x)$

7. Nature's moves: Non-strategic, can be random, according to a commonly known probability distribution

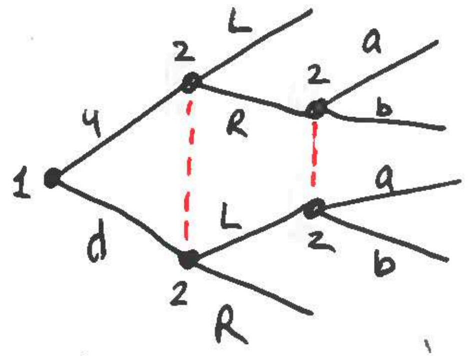
8. Payoffs: For each player i, $u_i : Z \to \mathbb{R}$ specifies player i's payoff $u_i(z)$ at each terminal node $z \in Z$.

Terminology

1. A game of perfect recall : Players don't forget anything they have observed or known.



Player 1 has imperfect recall



Player 2 has imperfect recall

2. A game of perfect information = "no hidden moves". Each info. set contains only one node. Players know the full history of previous moves.

→ Simultaneous-move games are games of imperfect information.

3. A game of <u>complete information</u>:
The structure of the game (in particular, the players' payoffs) is common knowledge.

<u>Strategies</u>

Let H_i be the collection of info. sets of player i:

$$H_i = \{h : i(h) = i\}$$

Let A_i be the set of actions of player i in the game tree:

$$A_i = \bigcup_{h \in H_i} A(h)$$

A <u>pure strategy</u> of player i is a function $s_i : H_i \to A_i$ with the property that $s_i(h) \in A(h)$ for all $h \in H_i$.

s_i is a <u>complete set of instructions</u> what to do at every node where player i needs to make a choice.

S_i = set of pure strategies of player i

$S = S_1 \times S_2 \times \ldots \times S_N$ = set of pure strategy profiles of all players

$s = (s_1, s_2, \ldots, s_N) \in S$ is a strategy profile.

Example

$i(x_0) = 1$

$i(x_1) = 2$

$i(x_2) = 1$

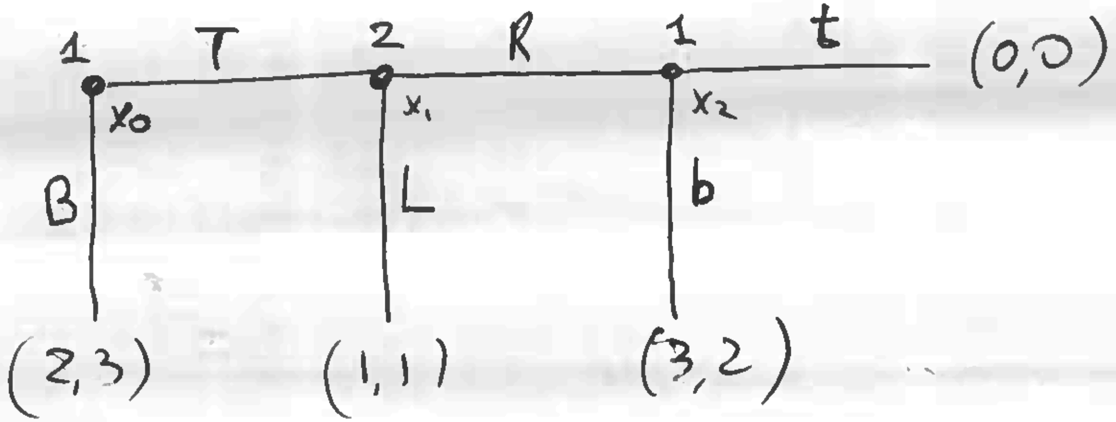
$A(x_0) = \{T, B\}$

$A(x_1) = \{L, R\}$

$A(x_2) = \{t, b\}$

$H_1 = \{x_0, x_2\}$

$H_2 = \{x_1\}$



$S_2 = \{L, R\}$

$S_1 = \{Tt, Tb, Bt, Bb\}$

B is not a strategy!

Outcomes

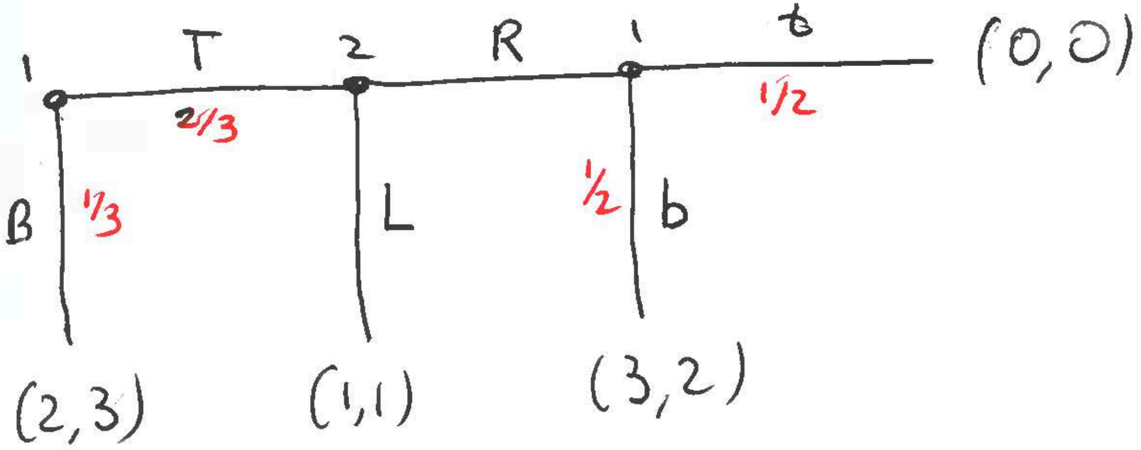
$o: S \to \Delta(Z)$ = the outcome of a strategy profile s

$o(s)[z]$ = the probability that terminal node z is reached when strategy profile s is played.

Expected payoffs:

$U_i : S \to \mathbb{R}$ where $U_i(s) = \sum_{z \in Z} o(s)[z] \, u_i(z)$

A <u>mixed strategy</u> $\sigma_i \in \Delta(S_i)$ is a probability distribution over pure strategies.



$S_1 = \{ Tt,\ Tb,\ Bt,\ Bb \}$

Example: $\sigma_1 \quad \frac{1}{3} \quad \frac{1}{3} \quad 0 \quad \frac{1}{3}$

$\sigma_1(Tt) = \sigma_1(Tb) = \sigma_1(Bb) = \frac{1}{3}$

$\sigma_1(Bt) = 0.$

$$\sigma_1(B) = \sigma_1(Bb) + \sigma_1(Bt) = \frac{1}{3} + 0 = \frac{1}{3}$$

$$\sigma_1(T) = \sigma_1(Tt) + \sigma_1(Tb) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

$$\sigma_1(b \mid T) = \frac{\sigma_1(Tb)}{\sigma_1(T)} = \frac{1/3}{2/3} = \frac{1}{2}$$

$$\sigma_1(t \mid T) = \frac{\sigma_1(Tt)}{\sigma_1(T)} = \frac{1/3}{2/3} = \frac{1}{2}$$

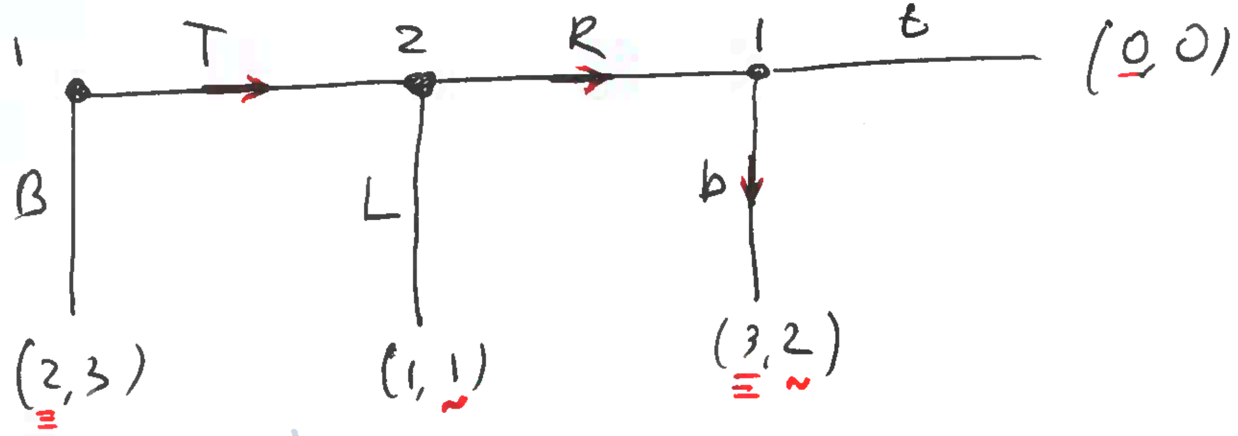
Bayes' rule

A <u>behavior strategy</u> $b_i \in \prod_{h \in H_i} \Delta(A(h_i))$ describes the probabilities of choosing actions in each of player i's information sets $h \in H_i$.

$b_i(h)[a]$ = the probability of choosing action $a \in A(h)$ in the info. set h

<u>Proposition</u> For games with <u>perfect recall</u>, restricting attention to behavior strategies is without loss of generality.

Backward Induction (BI)



Backward Induction Equilibrium (Tb, R)

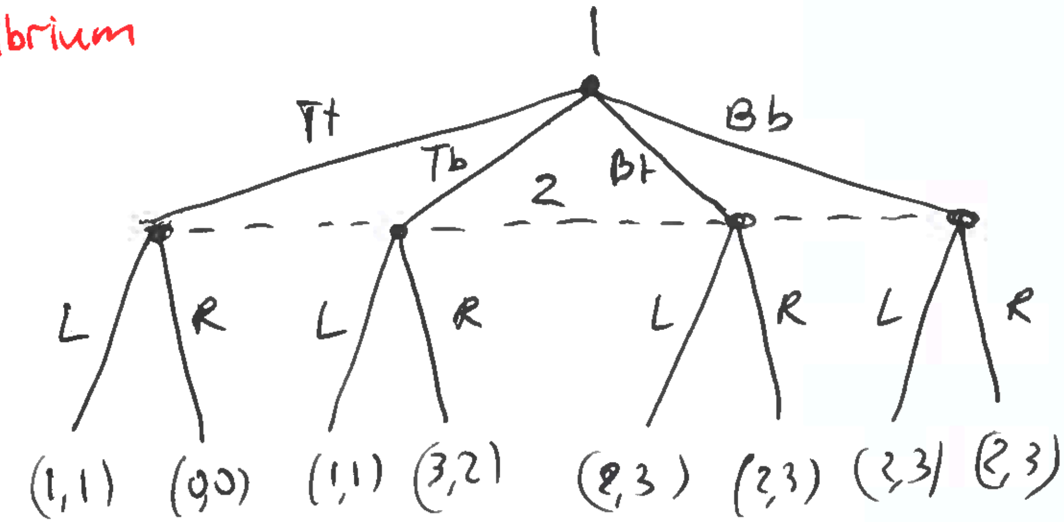
Game is solvable by BI = Game of perfect info.

$S_1 = \{Tt, Tb, Bt, Bb\}$ $S_2 = \{L, R\}$

	L	R
Tt	$1,1^*$	$0,0$
Tb	$1,1$	$3^*,2^*$
Bt	$2^*,3^*$	$2,3^*$
Bb	$2^*,3^*$	$2,3^*$

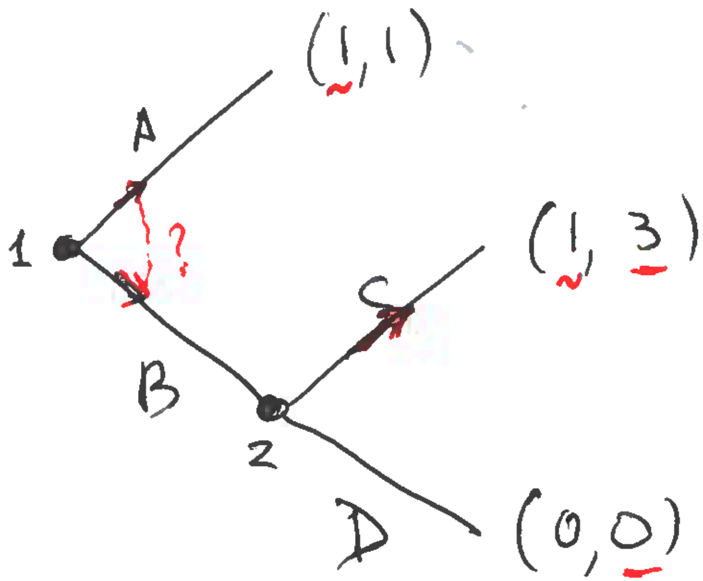
BI Equilibrium

NE



Proposition Every finite perfect info. game has at least one BI equilibrium in pure strategies. Moreover, this BI equilibrium is unique, provided players have no ties in payoffs.

Example: multiple BI eq:



Pure strategy BI equilibria

(A, C) and (B, C)

The set of BI equilibria:

$$\{ (\sigma_1(A), \sigma_1(B)), C \}$$

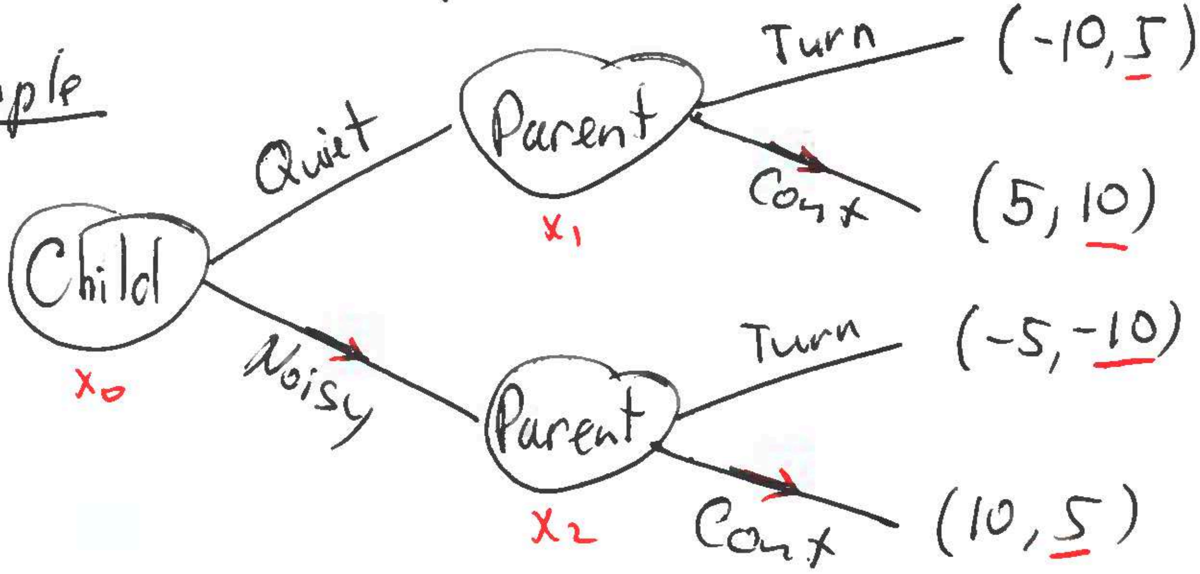
where $\sigma_1(A) + \sigma_1(B) = 1$

$\sigma_1(A), \sigma_1(B) \geq 0$

Nash Equilibrium (NE)

A NE of an extensive-form game is a NE of the corresponding normal (matrix)-form game.

Example



$S_1 = \{Q, N\}$

$S_2 = \{TT, TC, CT, CC\}$ (x_1 x_2)

BI equilibrium:

(N, CC)

	TT	TC	CT	CC
Q	-10, 5	-10, 5	5, 10	5, 10
N	-5, -10	10, 5	-5, -10	10, 5

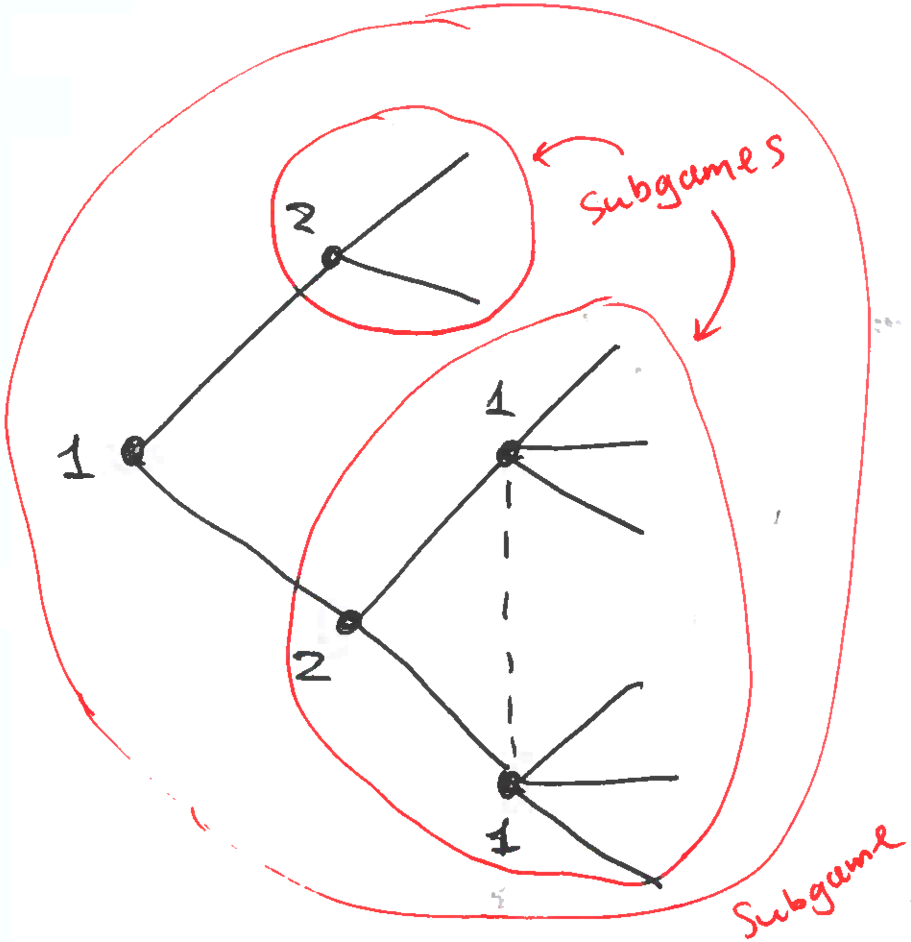
Non-credible threat (by the parent)

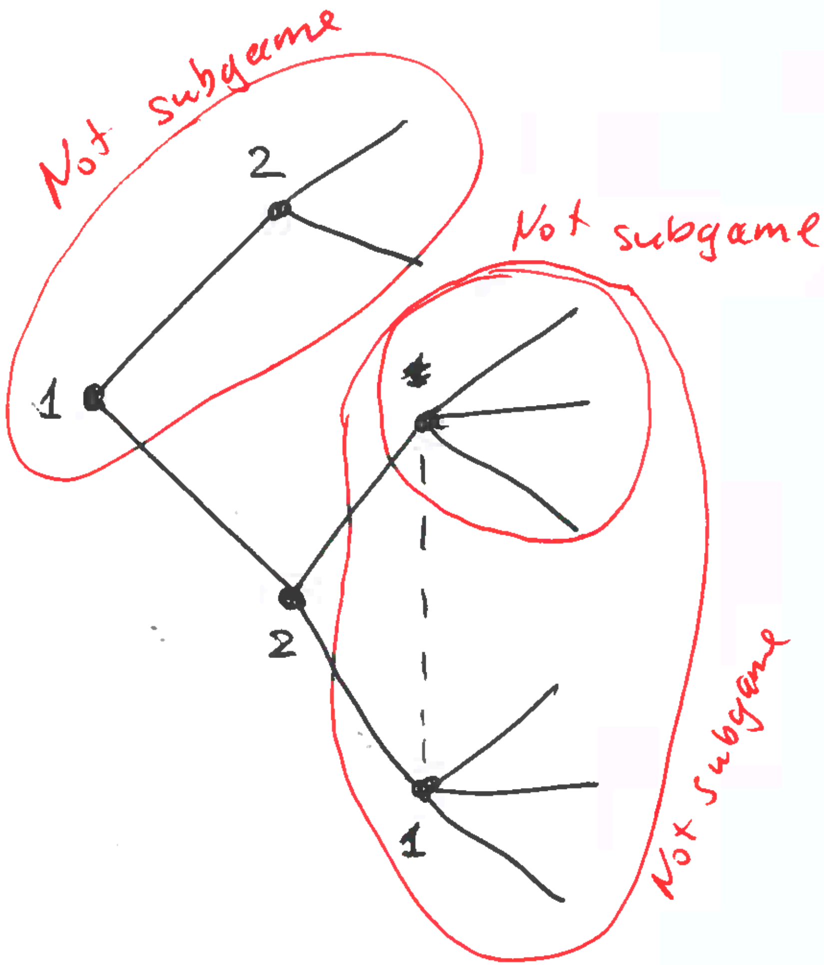
BI

Proposition: Every BI equilibrium is a NE, but the converse need not be true.

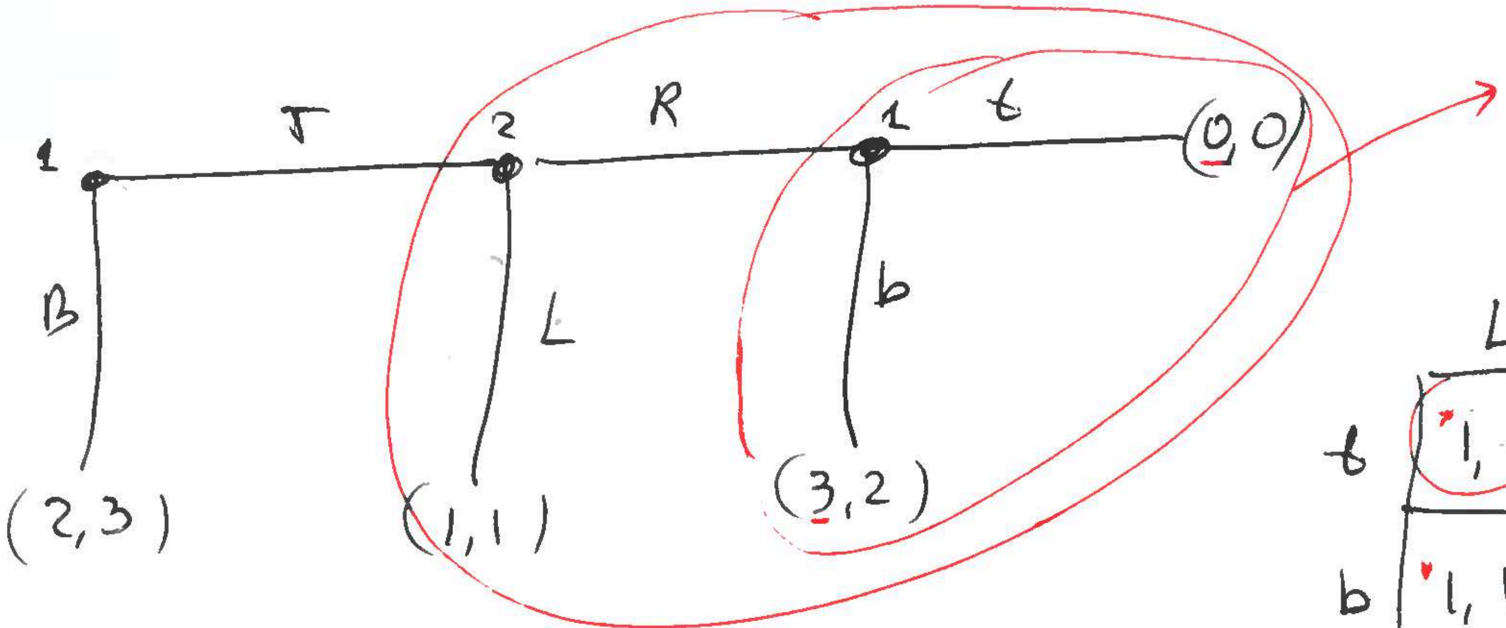
Subgame Perfect Equilibrium

Def A subgame is a subtree that does not break any information sets.





Def A subgame perfect eq. (SPE) is a profile of strategies such that their restriction to any subgame forms a NE of that subgame.



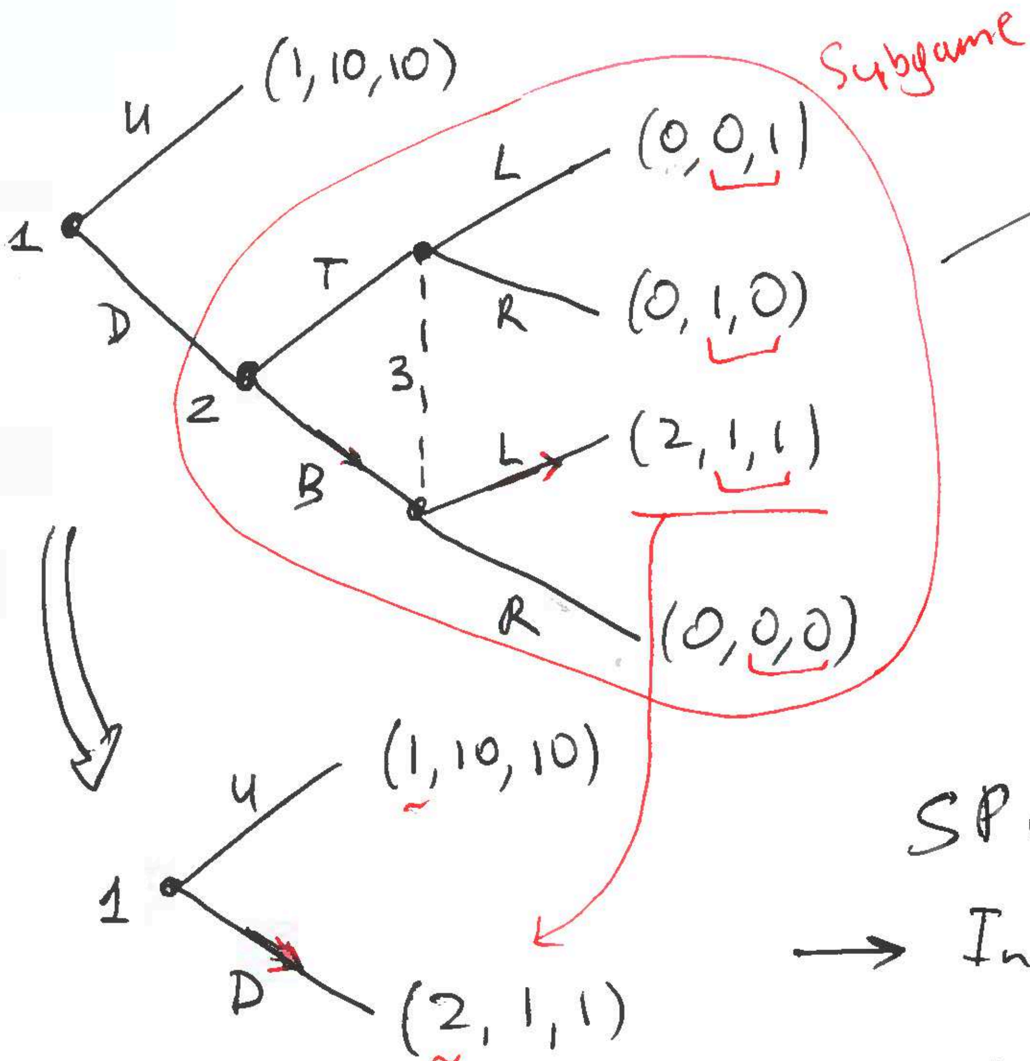
t	0
b	3

NE: b

	L	R
t	1, 1	0, 0
b	1, 1	3, 2

$$(Tb, R) \sim SPE = BI$$

3-player example



Player 3

Player 2	L	R
T	0,1	1,0
B	1,1	0,0

NE (B, L)
(ignoring player 1)

SPE : (D, B, L)

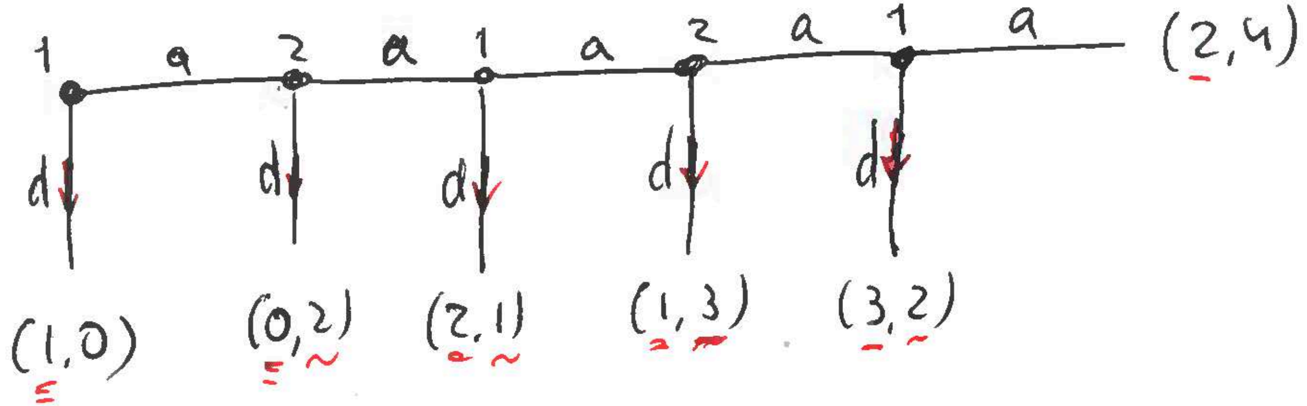
→ In games ~~with~~ of perfect info,
BI = SPE

→ In games that do not have any proper subgames,
SPE = NE of the whole game

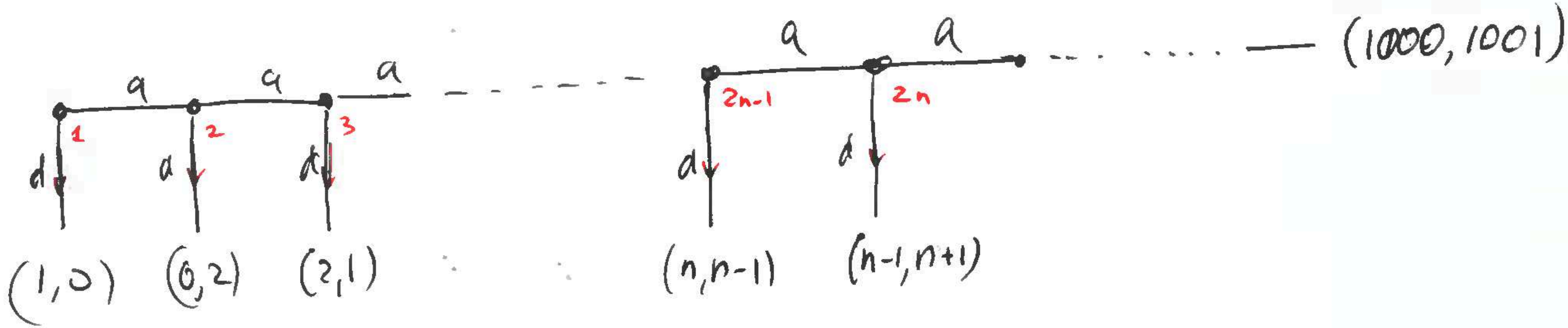
→ Every SPE is NE of the whole game.

Problems with BI and SPE

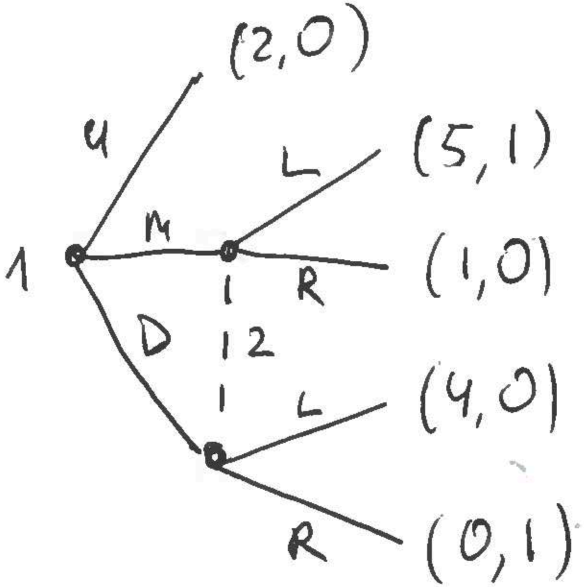
Centipede Game



BI: (ddd, dd)



SPE: Might Depend on seemingly irrelevant details.

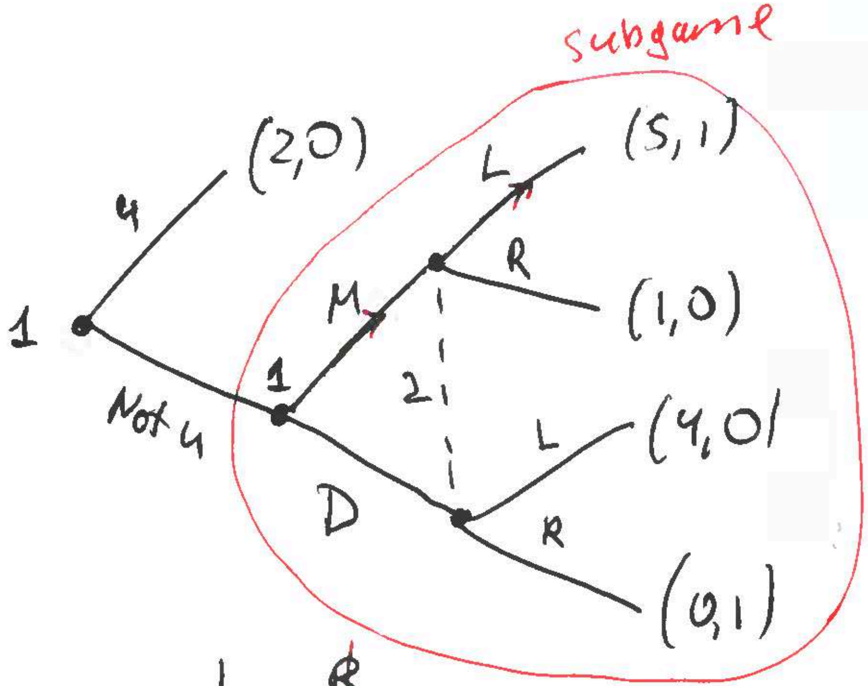


No proper subgames.

	L	R
U	2,0	2,0
M	5,1	1,0
D	4,0	0,1

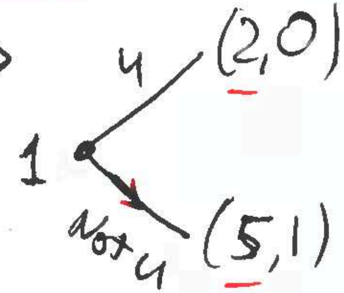
SPE

(M,L)
and
(U,R)



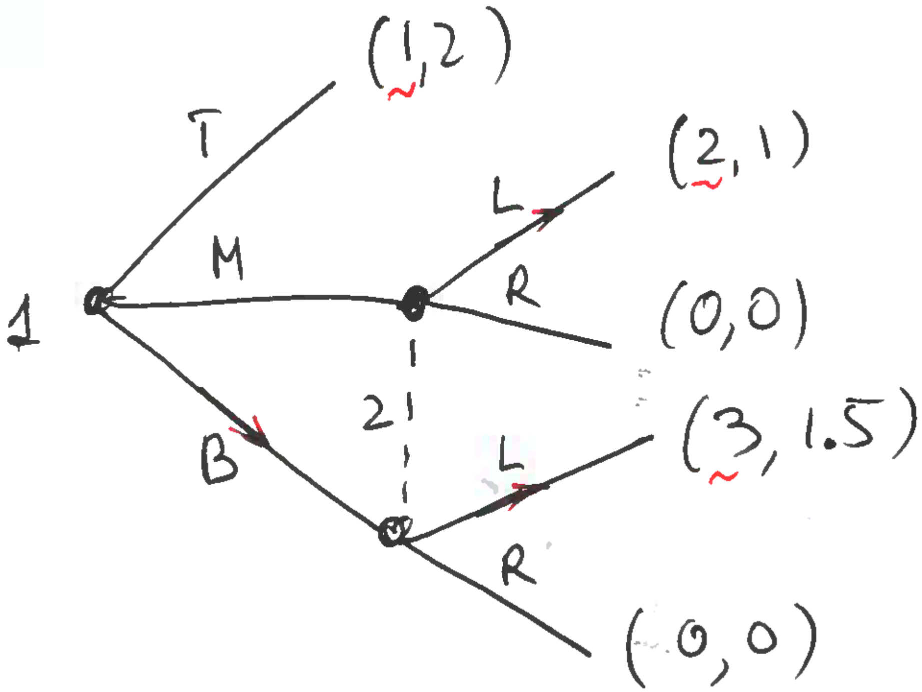
	L	~~R~~
M	5,1	1,0
~~D~~	4,0	0,1

⇒



SPE: (NotU,M ; L)

SPE Might not respect dominance



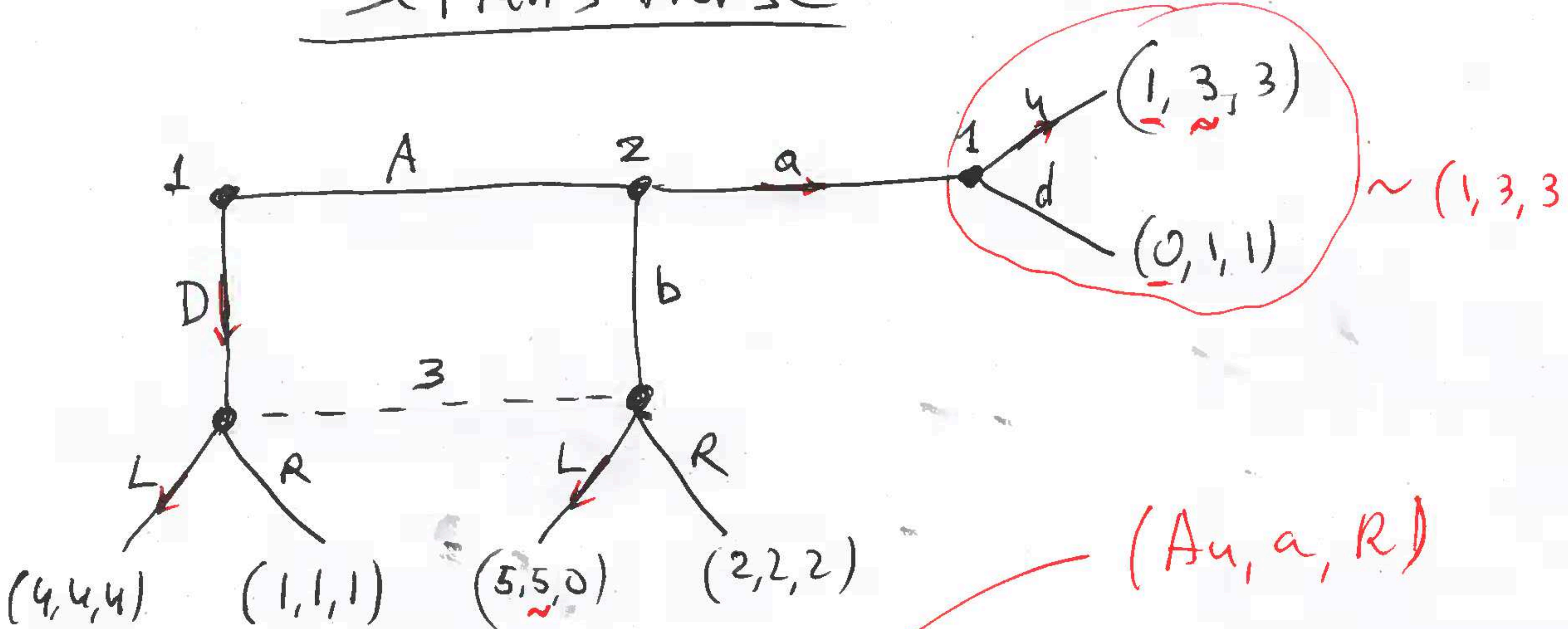
	L	R
T	1, 2*	*1, 2*
M	2, 1*	0, 0
B	*3, 1.5**	0, 0

SPE

L strictly dominates R

(B, L)

Selten's Horse



(Au, a, R)

L

	a	b
A	1, 3, 3	5, 5, 0
D	4, 4, 4	4, 4, 4

R

	a	b
A	1, 3, 3	2, 2, 2
D	1, 1, 1	1, 1, 1

3

(Du, a, L) ~ incansistent with BI Logic.