Mixed Actions

= randomizations over pure actions

A mixed action of player i is a probability distribution

$$\sigma_i = (\sigma_i(a_1), \underbrace{\sigma_i(a_2)}_{\text{probability to play action } a_2}, \dots, \sigma_i(a_K))$$

The support of a mixed action σ_i is

$$\mathrm{Supp}(\sigma_i) = \{ a_i \in A_i : \sigma_i(a_i) > 0 \}$$

Expected utility from playing mixed action σ_i given beliefs p_j, $i, j \in \{1, 2\}$, is

$$u_i(\sigma_i, p_j) = \sum_{a_i \in A_i} \sigma_i(a_i) \cdot u_i(a_i, p_j) = \sum_{a_i \in A_i,} \sum_{a_j \in A_j} \sigma_i(a_i)\, p_j(a_j)\, u_i(a_i, a_j)$$

belief p_j = mixed action p_j of player j

Consider player 1: $\max_{\sigma_1 \in \Delta(A_1)} u_1(\sigma_1, p_2)$

$$BR_1(p_2) = \underset{\sigma \in \Delta(A_1)}{\arg\max} \; u_1(\sigma_1, p_2)$$

Proposition A mixed action σ_1 is a BR to beliefs p_2, so $\sigma_1 \in BR(p_2)$, if and only if every pure action in the support of σ_1 is a BR to p_2. (In particular, every pure action in the support of σ_1 must have the same expected utility).

Interpretations of mixed action:

→ literally randomize (to confuse the opponent)

→ players have uncertainty (beliefs) about what others are doing.

→ population of players with different pure actions

Def. Action $a_1 \in A_1$ is strictly dominated by mixed action $\sigma_1 \in \Delta(A_1)$ if

$$u(a_1, a_2) < u(\sigma_1, a_2) \quad \text{for all } a_2 \in A_2$$

Example

Player 1

$$\begin{array}{c|c|c|} & L & R \\ \hline T & 3 & 0 \\ \hline M & 1 & 1 \\ \hline B & 0 & 3 \\ \hline \end{array} \left.\begin{array}{c} 1/2 \\ 0 \\ 1/2 \end{array}\right\} \sigma_1$$

M is strictly dominated by $\sigma_1 = (\underset{T}{\tfrac{1}{2}}, \underset{M}{0}, \underset{B}{\tfrac{1}{2}})$.

$$u_1(\sigma_1, L) = \tfrac{1}{2} \cdot 3 + 0 \cdot 1 + \tfrac{1}{2} \cdot 0 = 1.5 > u_1(M, L) = 1$$

$$u_1(\sigma_1, R) = \tfrac{1}{2} \cdot 0 + 0 \cdot 1 + \tfrac{1}{2} \cdot 3 = 1.5 > u_1(M, R) = 1$$

Proposition In 2-player games, an action $a \in A_1$ is strictly dominated (possibly by a mixed action) if and only if a is NBR.

Iterative Deletion of Dominated Actions

strictly

$$\mathrm{ID}\begin{pmatrix} S \\ W \end{pmatrix}\mathrm{DA}$$

weakly

Ex 1

		Rain God	
		Sun	~~Rain~~
Player 1	No umb.	$\underset{\sim}{5}, \underline{3}$	~~$0, \underline{0}$~~
	~~Umb.~~	~~$\underset{\sim}{1}, \underline{5}$~~	~~$3, \underline{2}$~~

① "Sun" strictly dominates "Rain"
→ delete "Rain"

② "No umb" strictly dominates "Umb"
→ delete "Umb"

Ex 2

		Pl. 2	
		L	~~R~~
Pl. 1	U	$\underset{\sim}{8}, \underline{10}$	~~$-1000, \underline{9}$~~
	~~D~~	~~$\underset{\sim}{7}, \underline{6}$~~	~~$6, \underline{5}$~~

① L strictly dominates R
→ delete R

② U strictly dominates D
→ delete D

Common knowledge (of rationality, of payoffs)

An event is common knowledge (CK) if all players know that it is true, and all know that all know that it is true, and so on.

Example $\mathrm{ID}\underline{\mathrm{W}}\mathrm{DA}$ ← weakly

$$\begin{array}{c|c|c|c} & D & E & F \\ \hline A & 1,1 & 1,1 & 2,1 \\ \hline B & 1,1 & 0,0 & 3,1 \\ \hline C & 1,2 & 1,3 & 1,1 \end{array}$$

Case 1 Start with Pl. 1:

① A weakly dominates C → delete C

② D weakly dominates E → delete E

③ B — " — " — A → delete A

$$\begin{array}{c|c|c} & D & F \\ \hline B & 1,1 & 3,1 \end{array}$$

⇒ (B, D) and (B, F)

Case 2 Start with Pl 2:

① D weakly dominates F → delete F

② A — " " — B → delete B

③ E — " — D → delete D

$$\begin{array}{c|c} & E \\ \hline A & 1,1 \\ \hline C & 1,3 \end{array}$$

⇒ (A, E) and (C, E)

IDSDA → order of deletion makes no difference

IDWDA → order can matter.

A game is dominance solvable if the procedure of IDSDA (or the procedure of IDWDA regardless of the order of deletion) results in a unique outcome.

Rationalizability

An action is rationalizable if it cannot be (iteratevely) deleted as NBR.

(the same as IDSDA in 2-player games)

Nash Equilibrium

An action profile $\sigma^* = (\sigma_1^*, \sigma_2^*, \dots, \sigma_N^*)$ is a Nash equilibrium if for each player i

$$u_i(\underset{\sim}{\sigma_i^*}, \underbrace{\sigma_{-i}^*}_{\uparrow \text{ the other players}}) \geq u(\underset{\sim}{a_i}, \sigma_{-i}^*) \quad \text{for every } a_i \in A_i$$

That is, $\sigma_i^* \in BR(\sigma_{-i}^*)$ for each player i.

= "mutual best response".

		Rain God	
		Sun	Rain
Pl. 1	No umb	$^{*}5, 3^{*}$	0, 0
	Umb	$1, 5^{*}$	$^{*}3, 2$

NE.

Proposition If $\sigma^* = (\sigma_1^*, \sigma_2^*)$ is a NE, then every action in its support

$$\begin{cases} \text{survives IDSDA} \\ \text{survives ID } \underline{\text{NBR}} \text{ A} \end{cases}$$

(Never-best-response)

	W	B	
W	$^*2, 1^*$	0, 0	q
B	0, 0	$^*1, 2^*$	1-q
	p	1-p	

NE
(W, W)
(B, B)

$$u_1(W, p) = 2\cdot p + 0\cdot(1-p) = 2p$$
$$u_1(B, p) = 0\cdot p + 1\cdot(1-p) = 1-p$$

For player 1 to mix between W and B, the expected payoffs have to be the same:

$$2p = 1-p \Rightarrow \boxed{p = 1/3}$$

$$u_2(W, q) = 1\cdot q + 0\cdot(1-q) = q$$
$$u_2(B, q) = 0q + 2(1-q) = 2-2q$$

$$\boxed{q = 2-2q} \Rightarrow \boxed{q = 2/3}$$

<u>Mixed strategy NE:</u>

$$\left(\underbrace{\left(\underset{W}{\overset{q}{\tfrac{2}{3}}}, \underset{B}{\overset{1-q}{\tfrac{1}{3}}}\right)}_{\text{Player 1}}, \underbrace{\left(\underset{W}{\overset{p}{\tfrac{1}{3}}}, \underset{B}{\overset{1-p}{\tfrac{2}{3}}}\right)}_{\text{Player 2}}\right)$$

BR correspondence and NE

	W	B	
W	2,1	0,0	q
B	0,0	1,2	1-q
	p	1-p	

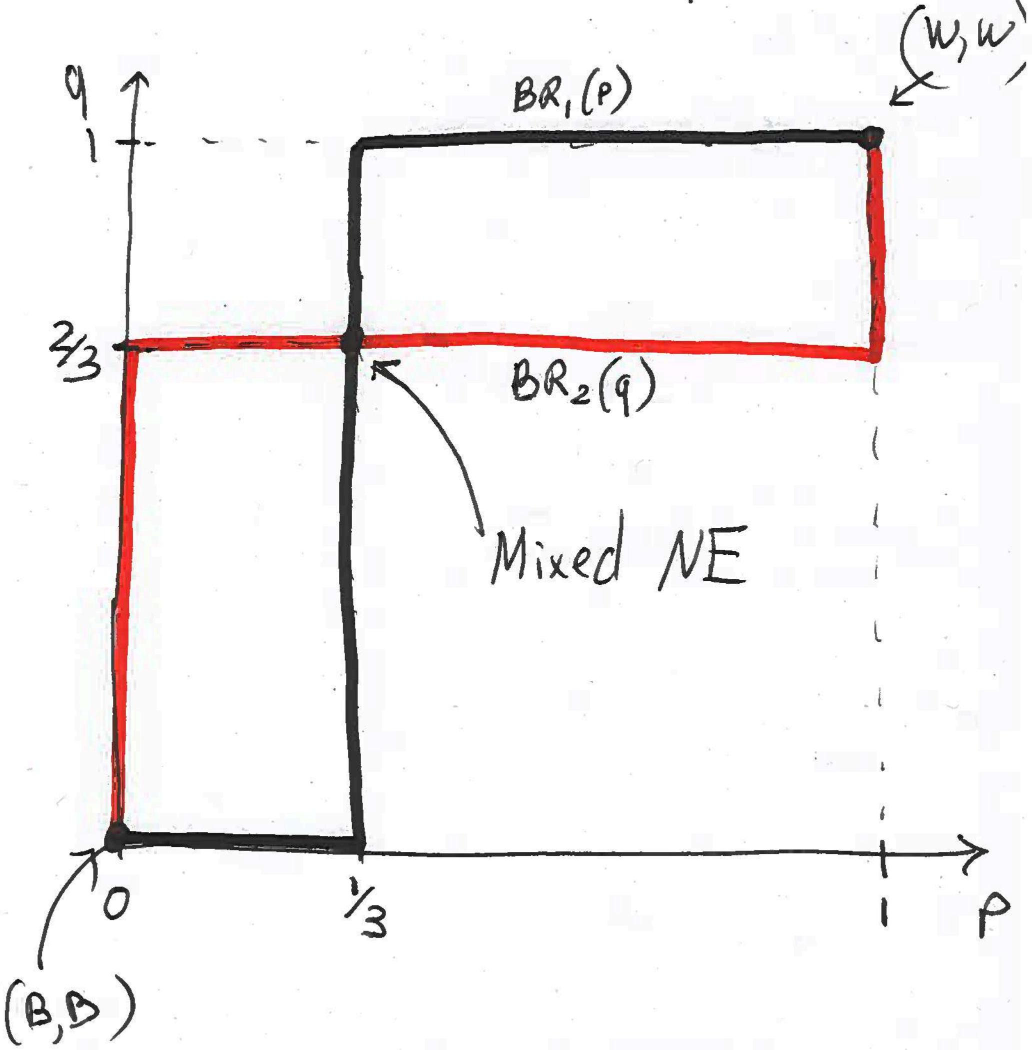
$u_1(W,p) = 2p, \quad u_1(B,p) = 1-p.$

$$BR_1(p) = \begin{cases} W, & p > \frac{1}{3} \\ \text{anything} & p = \frac{1}{3} \\ B, & p < \frac{1}{3} \end{cases}$$

$$BR_1(p) = \begin{cases} q = 1, & p > \frac{1}{3} \\ q \in [0,1], & p = \frac{1}{3} \\ q = 0, & p < \frac{1}{3} \end{cases}$$

$u_2(W,q) = q, \quad u_2(B,q) = 2-2q$

$$BR_2(q) = \begin{cases} p = 1, & q > 2/3 \\ p \in [0,1], & q = 2/3 \\ p = 0, & q < 2/3 \end{cases}$$

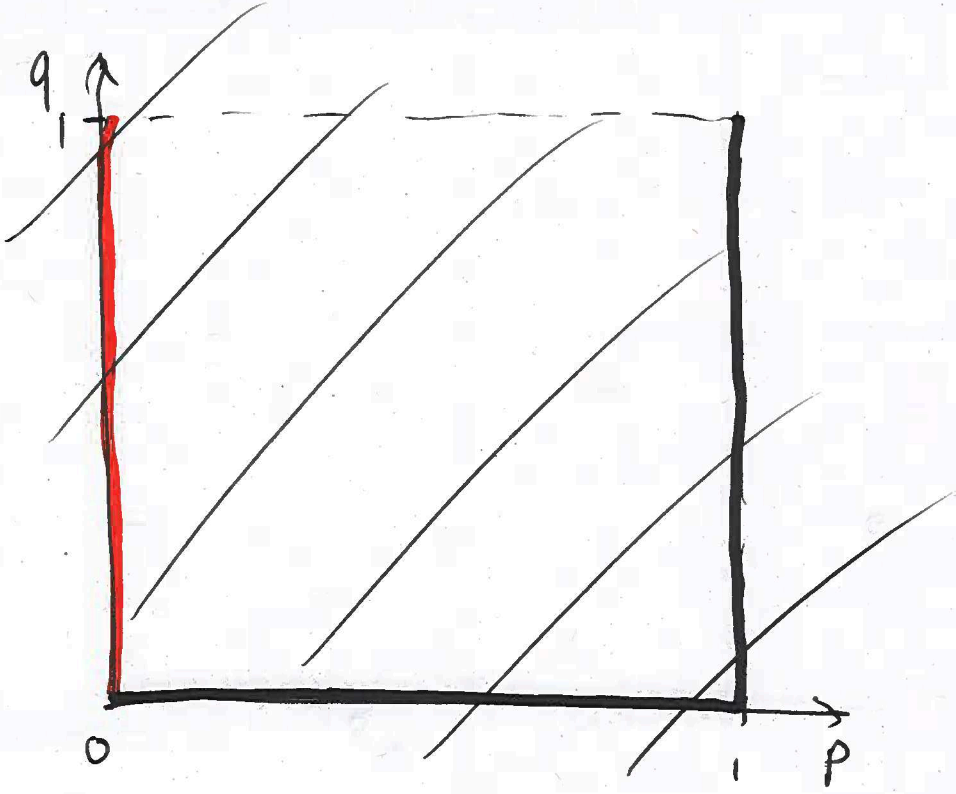


$$\begin{array}{c|c|c|c} & L & R & \\ \hline T & {}^*10, 0 & \boxed{{}^*5, 2^*} & q \\ \hline B & \boxed{{}^*10, 11^*} & 2, 11^* & 1-q \\ \hline & p & 1-p & \end{array}$$

$$\left.\begin{array}{l} ① \ T \text{ weakly dominates } B \\ ② \ R \text{ — '' — '' } L \end{array}\right\} \Rightarrow (T,R)$$

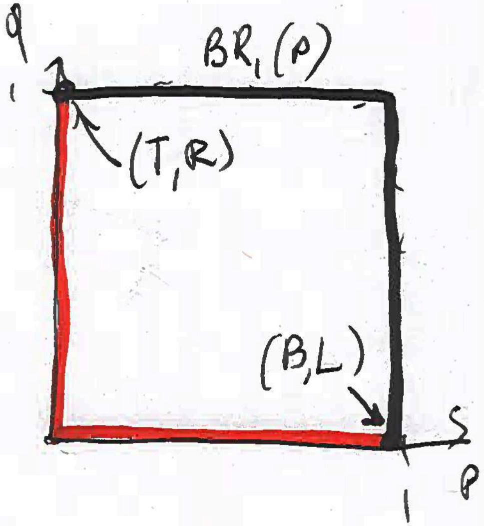
$$\left.\begin{array}{l} ① \ R \text{ — '' — '' } L \\ ② \ T \text{ — '' — '' } B \end{array}\right\} \Rightarrow (T,R)$$

Dominance soluable!



$$\left.\begin{array}{l} u_1(T,p) = 10p + 5(1-p) \\ u_1(B,p) = 10p + 2(1-p) \end{array}\right\} \quad BR_1(p) = \begin{cases} q=1, & p<1 \\ q\in[0,1], & p=1 \end{cases}$$

$$\left.\begin{array}{l} u_2(L,q) = 0\cdot q + 11(1-q) \\ u_2(R,q) = 2\cdot q + 11(1-q) \end{array}\right\} \quad BR_2(q) = \begin{cases} p=0, & q>0 \\ p\in[0,1], & q=0. \end{cases}$$



Prisoner's Dilemma

	C	D
C	4,4	0,5
D	5,0	1,1

D strictly dominates C

Coordination game

	A	B
A	*10,10*	0,0
B	0,0	*1,1*

$$\begin{array}{c|c|c|c}
 & L & R & \\ \hline
T & {}^{*}3,0 & 0,3^{*} & q \\ \hline
\cancel{M} & \cancel{1,1^{*}} & \cancel{1,1^{*}} & 0 \\ \hline
B & 0,3^{*} & {}^{*}3,0 & 1-q \\ \hline
 & p & 1-p &
\end{array}$$

No pure NE.

$$u_1(T,p) = 3p + 0(1-p)$$

$$u_1(B,p) = 0.p + 3(1-p)$$

$$3p = 3(1-p)$$

$$\boxed{p = 1/2}$$

$$u_2(L,q) = 0\cdot q + 3(1-q)$$

$$u_2(R,q) = 3\cdot q + 0\cdot(1-q)$$

$$3(1-q) = 3q, \quad \boxed{q = 1/2}$$

NE in mixed actions

$$\left(\left(\underset{T}{\tfrac{1}{2}}, \underset{M}{0}, \underset{B}{\tfrac{1}{2}} \right), \left(\underset{L}{\tfrac{1}{2}}, \underset{R}{\tfrac{1}{2}} \right) \right)$$

Example 3 players

Box 1 (Player 1 rows, Player 2 columns)

	X	Y
A	*1, *1, 0	*1, 0, 1*
B	*1, *1, *1	0, *1, *1

Box 2 (Player 1 rows, Player 2 columns)

	X	Y
A	*1, 0, 1*	*1, *1, 0
B	*1, *1, 0	0, *1, 0

Box 1 ⟷ Box 2: Player 3

NE in pure actions
(B, X, Box 1)

A weakly dominates B
delete B

A

Player 3 (columns), Player 2 (rows)

	Box 1	Box 2	
X	1, 0	0, 1	1/2
Y	0, 1	1, 0	1/2
	1/2	1/2	

NE → (B, X, Box 1)

NE → $\left(A, \left(\underset{X}{\tfrac{1}{2}}, \underset{Y}{\tfrac{1}{2}}\right), \left(\underset{Box1}{\tfrac{1}{2}}, \underset{Box2}{\tfrac{1}{2}}\right)\right)$

Divide the Dollar

$A_1 = A_2 = \mathbb{R}_+$ $\quad$ $u_i(a_i, a_j) = \begin{cases} a_i, & a_i + a_j \le 1 \\ 0, & a_i + a_j > 1 \end{cases}$

$a_j = x < 1 \Rightarrow BR_i(x) = \{1 - x\}$

$a_j = x \geq 1 \Rightarrow BR_i(x) = \mathbb{R}_+$

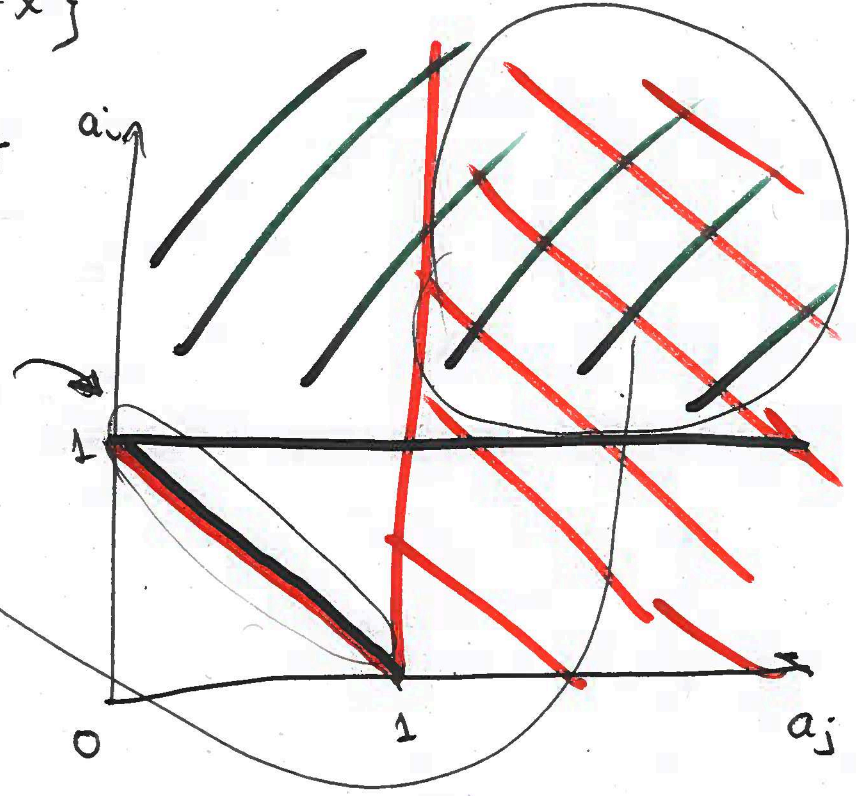
NE: any (a_i, a_j):

$a_i + a_j = 1$ $\quad$ ex. $(0.4, 0.6)$

any (a_i, a_j):

$\min\{a_i, a_j\} \geq 1$

ex. $(2, 1)$



NE in weakly dominated actions.