Mixed Actions

= randomizations over pure actions

A mixed action of player i is a probability distribution

$$\sigma_i = (\sigma_i(a_1), \underline{\sigma_i(a_2)}, \dots, \sigma_i(a_K))$$

↖ probability to play a_2.

The <u>support</u> of a mixed action σ_i is

$$\text{Supp}(\sigma_i) = \{ a_i \in A_i : \sigma_i(a_i) > 0 \}$$

Expected utility from playing mixed action σ_i given beliefs p_j about player j, $i, j \in \{1,2\}$

is

$$u_i(\sigma_i, p_j) = \sum_{a_i \in A_i} \sigma_i(a_i) u_i(a_i, p_j) = \sum_{a_i \in A_i} \sum_{a_j \in A_j} \sigma_i(a_i) p_j(a_j) u_i(a_i, a_j)$$

belief p_j = mixed action of player j

Consider player 1: $\max_{\sigma_1 \in \Delta(A_1)} u_1(\sigma_1, p_2)$

$$BR_1(p_2) = \underset{\sigma_1 \in \Delta(A_1)}{\arg\max}\ u_1(\sigma_1, p_2)$$

<u>Proposition</u> A mixed action σ_1 ~~$\in BR$ [illegible]~~ is a best response to beliefs p_2, so $\sigma_1 \in BR(p_2)$, if and only if every pure action in the support of σ_1 is itself a best response to p_2.

(In particular, every pure action in the support of σ_1 must have the same expected utility)

<u>Interpretations of mixed actions</u>

→ literally randomise (to confuse the opponent)

→ players have uncertainty (beliefs) about what the others are doing.

→ population of players with different pure actions

Def Action $a_1 \in A_1$ is <u>strictly dominated</u> by a mixed action $\sigma_1 \in \Delta(A_1)$ if

$$u_1(a_1, a_2) < u_1(\sigma_1, a_2) \quad \text{for all } a_2 \in A_2.$$

Example

Player 1

$$\begin{array}{c|c|c|c} & L & R & \\ \hline T & 3 & 0 & \frac{1}{2} \\ \hline M & 1 & 1 & 0 \\ \hline B & 0 & 3 & \frac{1}{2} \end{array} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\} \sigma_1$$

M is strictly dominated by $\sigma_1 = (\underset{T}{\tfrac{1}{2}}, \underset{M}{0}, \underset{B}{\tfrac{1}{2}})$

$$u_1(M, L) = 1 < u_1(\sigma_1, L) = \tfrac{1}{2}\cdot 3 + 0\cdot 1 + \tfrac{1}{2}\cdot 0 = 1.5$$

$$u_1(M, R) = 1 < u_1(\sigma_1, R) = \tfrac{1}{2}\cdot 0 + 0\cdot 1 + \tfrac{1}{2}\cdot 3 = 1.5$$

Proposition In 2-player games, an action $a_1 \in A_1$ is strictly dominated (possibly, by a mixed action) if and only if a_1 is NBR.

Iterative Deletion of Dominated Actions

$ID\binom{S}{W}DA$ strictly weakly

Ex 1

Rain God

Player 1

	Sun	~~Rain~~
No umb	5, 3	~~0, 0~~
Umb	~~1, 5~~	~~3, 2~~

① "Sun" strictly dominates "Rain"
→ delete "Rain"

② "No umb" strictly dominates "Umb"
→ delete "Umb"

Ex 2

Pl. 2

Pl. 1

	L	~~R~~
U	8, 10	~~-1000, 9~~
D	~~7, 6~~	~~6, 5~~

① L strictly dominates R
→ delete R

② U strictly dominates D
→ delete D

Common Knowledge (of rationality, of payoffs)

An event is <u>common knowledge</u> (CK) if all players know that it is true, and all know that all know that it is true, and so on.

IDSDA → order of deletion makes no difference

IDWDA → order may matter

A game is dominance soluable if the procedure of IDSDA (or the procedure of IDWDA regardless of the order of deletion) results in a unique outcome.

Rationalizability

An action is rationalizable if it cannot be (iteratively) deleted as NBR.

(the same as IDSDA in 2-player games).

Example IDWDA

↖ weakly

	D	E	F
A	1,1	1,1	2,1
B	1,1	0,0	3,1
C	1,2	1,3	1,1

Case 1 Start with Pl. 1:

① A weakly dominates C
→ delete C

② D ~~w~~ — " — " — E
→ delete E

③ B — " — " A
→ delete A

	D	F
B	1,1	3,1

⇒ (B, D) or (B, F)

Case 2 Start with Pl. 2:

① D weakly dominates F → delete F

② A — " — " — " B → delete B

③ E — " — " — " D → delete D

⇒ (A, E) or (C, E)

	E
A	1,1
C	1,3

Nash Equilibrium

An action profile $\sigma^* = (\sigma_1^*, \sigma_2^*, \dots, \sigma_N^*)$ is a Nash Equilibrium if for each player i

$$u_i(\sigma_i^*, \underbrace{\sigma_{-i}^*}_{\uparrow \text{actions of players other than } i}) \geq u_i(a_i, \sigma_{-i}^*) \quad \text{for all } a_i \in A_i.$$

That is, $\sigma_i^* \in BR(\sigma_{-i}^*)$ for each player i.

= "mutual best response"

Rain God

Player 1

	Sun	Rain
No Umb	$^*5, 3^*$	0, 0
Umb	$1, 5^*$	$^*3, 2$

NE

Proposition If $\sigma^* = (\sigma_1^*, \sigma_2^*)$ is a NE, then every action in its support:

- → survives IDSDA
- → is rationalizable
- → may or may not survive IDWDA

Multiple Nash Equilibria

Pl.2

Pl.1

$$\begin{array}{c|c|c|c}
 & W & B & \\ \hline
W & \boxed{{}^*2,1^*} & 0,0 & q \\ \hline
B & 0,0 & \boxed{{}^*1,2^*} & 1-q \\ \hline
 & p & 1-p &
\end{array}$$

NE

$$u_1(W,p) = 2\cdot p + 0\cdot(1-p) = 2p$$
$$u_1(B,p) = 0\cdot p + 1\cdot(1-p) = 1-p$$

For player 1 to mix between W and B, the expected payoffs have to be equal:

$$2p = 1-p \Rightarrow \boxed{p = 1/3}$$

$$u_2(W,q) = 1\cdot q + 0\cdot(1-q) = q$$
$$u_2(B,q) = 0\cdot q + 2\cdot(1-q) = 2-2q$$

$$q = 2-2q \Longrightarrow \boxed{q = 2/3}.$$

Mixed ~~strategy~~ action NE: $\left(\underbrace{\left(\underset{W}{\overset{q}{\frac{2}{3}}}, \underset{B}{\overset{1-q}{\frac{1}{3}}}\right)}_{\text{Player 1}}, \underbrace{\left(\underset{W}{\overset{p}{\frac{1}{3}}}, \underset{B}{\overset{1-p}{\frac{2}{3}}}\right)}_{\text{Player 2}}\right)$

Pure NE: (W,W), (B,B)

BR correspondence and NE

$$\begin{array}{c|c|c|c} & W & B & \\ \hline W & 2,1 & 0,0 & q \\ \hline B & 0,0 & 1,2 & 1-q \\ \hline & p & 1-p & \end{array}$$

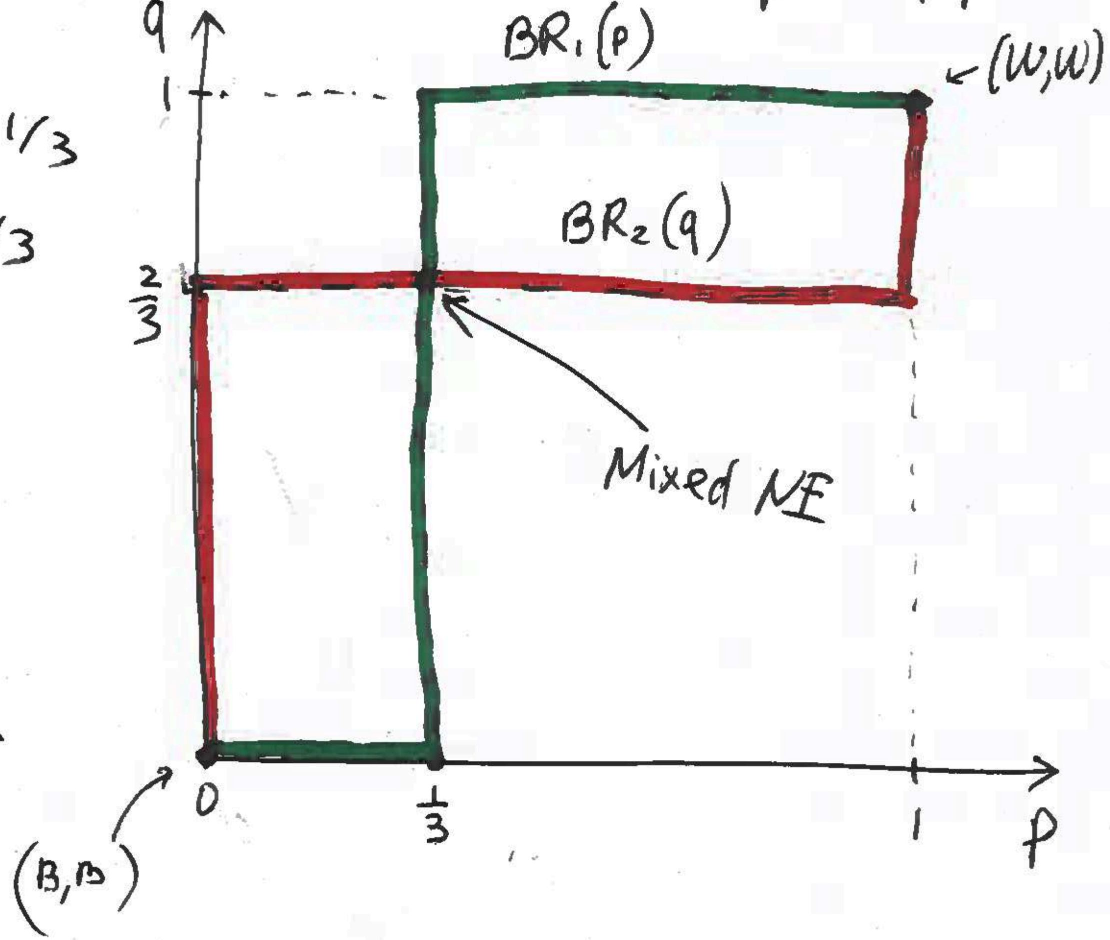
$u_1(W,p) = 2p \qquad u_1(B,p) = 1-p$

$$BR_1(p) = \begin{cases} W, & \text{if } p > \frac{1}{3} \\ \text{anything} & \text{if } p = 1/3 \\ B, & \text{if } p < \frac{1}{3} \end{cases}$$

$$BR_1(p) = \begin{cases} q = 1, & p > 1/3 \\ q \in [0,1], & p = 1/3 \\ q = 0, & p < 1/3 \end{cases}$$

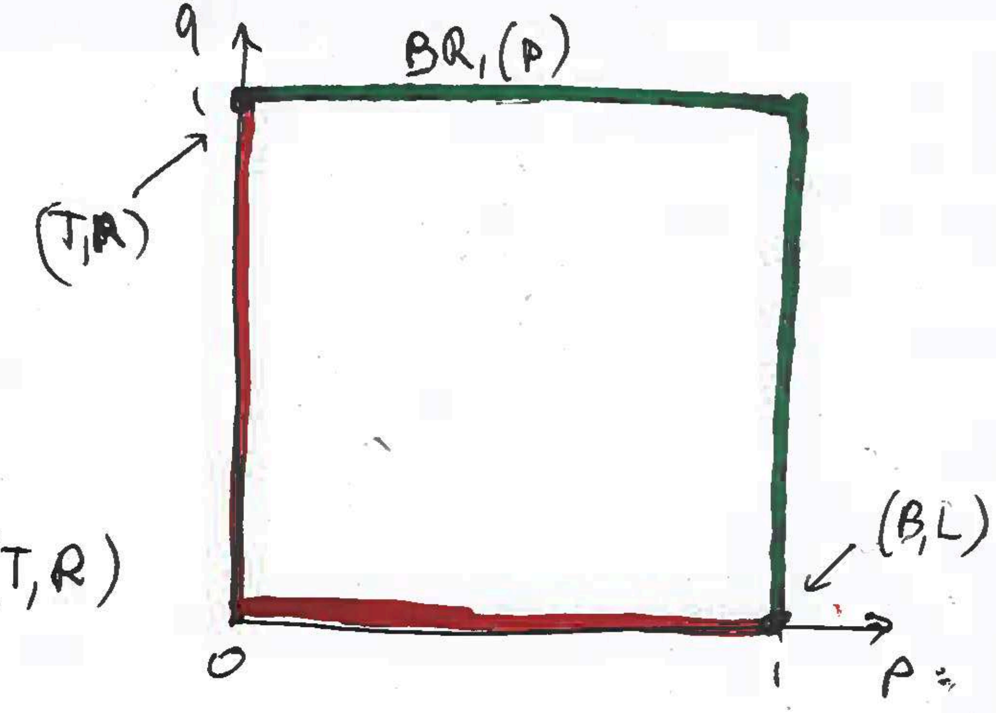
$u_2(W,q) = q \qquad u_2(B,q) = 2-2q$

$$BR_2(q) = \begin{cases} p = 1, & \text{if } q > \frac{2}{3} \\ p \in [0,1], & \text{if } q = \frac{2}{3} \\ p = 0, & \text{if } q < \frac{2}{3}. \end{cases}$$



		Pl. 2		
		L	R	
Pl.1	T	$^{*}10,0$	$^{*}5,2^{*}$	q
	B	$^{*}10,11^{*}$	$2,11^{*}$	$1-q$
		p	$1-p$	

NE



① T weakly dominates B
② R — " — L
} ⇒ (T,R)

① R — " — L
② T — " — B
} ⇒ (T,R) Dominance soluable!

Notice: (B,L) is deleted!

$$\left.\begin{aligned} u_1(T,p) &= 10\cdot p + 5(1-p) \\ u_1(B,p) &= 10\cdot p + 2(1-p) \end{aligned}\right\} \Rightarrow BR_1(p) = \begin{cases} q=1, & \text{if } p<1 \\ q\in[0,1] & \text{if } p=1 \end{cases}$$

$$\left.\begin{aligned} u_2(L,q) &= 0\cdot q + 11(1-q) \\ u_2(R,q) &= 2q + 11(1-q) \end{aligned}\right\} \Rightarrow BR_2(q) = \begin{cases} p=0, & \text{if } q>0 \\ p\in[0,1] & \text{if } q=0 \end{cases}$$

Prisoner's Dilemma

	C	D
C	4,4	$0,5^*$
D	^{*5,0}	$^*1,1^*$

D strictly dominates C

Coordination Game

	A	B
A	10,10	0,0
B	0,0	1,1

	L	R	
T	*3,0	0,3*	q
~~M~~	~~1,1*~~	~~1,1*~~	
B	0,3*	*3,0	1-q
	p	1-p	

No pure NE

NBR

$u_1(T,p) = 3\cdot p + 0(1-p) = 3p$

$u_1(B,p) = 0\cdot p + 3(1-p)$

$3p = 3(1-p)$

$\boxed{p = 1/2}$

Similarly for u_2: $\boxed{q = 1/2}$

NE in mixed actions:

$$\left(\left(\overset{q}{\underset{T}{\frac{1}{2}}}, \underset{M}{0}, \overset{1-q}{\underset{B}{\frac{1}{2}}}\right), \left(\overset{p}{\underset{L}{\frac{1}{2}}}, \overset{1-p}{\underset{R}{\frac{1}{2}}}\right)\right)$$

Example : 3 players

Box 1

Player 2

Player 1	X	Y
A	*1, *1, 0	*1, 0, 1*
B	*1, *1, *1	0, *1, *1

NE

(B, X, Box1)

Box 2

Player 2

	X	Y
A	*1, 0, 1*	*1, *1, 0
B	*1, *1, 0	0, *1, 0

Player 3

A weakly dominates B

→ delete B

Player 3

Ⓐ

Player 2	Box 1	Box 2	
X	1, 0	0, 1*	1/2
Y	0, 1*	*1, 0	1/2
	1/2	1/2	

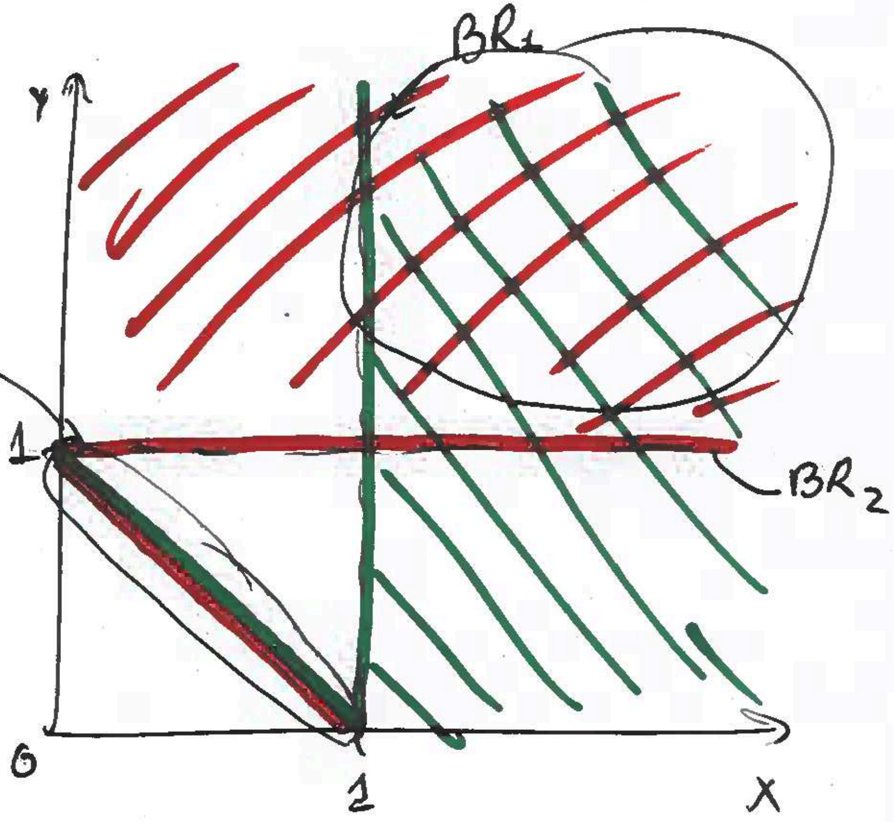
NE → (B, X, Box 1)

NE → $\left(A, \left(\underset{X}{\frac{1}{2}}, \underset{Y}{\frac{1}{2}}\right), \left(\underset{Box1}{\frac{1}{2}}, \underset{Box2}{\frac{1}{2}}\right)\right)$

Divide the Dollar

$A_1 = A_2 = \mathbb{R}_+$ $\quad u_i(a_i, a_j) = \begin{cases} a_i, & \text{if } a_i + a_j \leq 1 \\ 0, & \text{if } a_i + a_j > 1 \end{cases}$

$BR_i(x) = \begin{cases} 1 - x, & \text{if } x < 1 \\ \mathbb{R}_+, & \text{if } x \geqslant 1 \end{cases}$



NE

① any (a_i, a_j):

$a_i + a_j = 1, \quad a_i, a_j \in [0, 1]$.

② $\min\{a_i, a_j\} \geqslant 1$

↑ NE in weakly dominated actions