Repeated Games

Example

$$G = \begin{array}{c|ccc} & F & B & C \\ \hline F & 2,1 & 0,0 & 6,0 \\ B & 0,0 & 1,2 & 0,0 \\ C & 0,0 & 0,0 & 5,5 \end{array}$$

"stage game"

NE: (F, F)

(B, B)

$$\Big(\underbrace{\big(\tfrac{2}{3}, \tfrac{1}{3}, 0\big)}_{\text{Pl. 1}}, \underbrace{\big(\tfrac{1}{3}, \tfrac{2}{3}, 0\big)}_{\text{Pl. 2}} \Big)$$

F B C   F B C

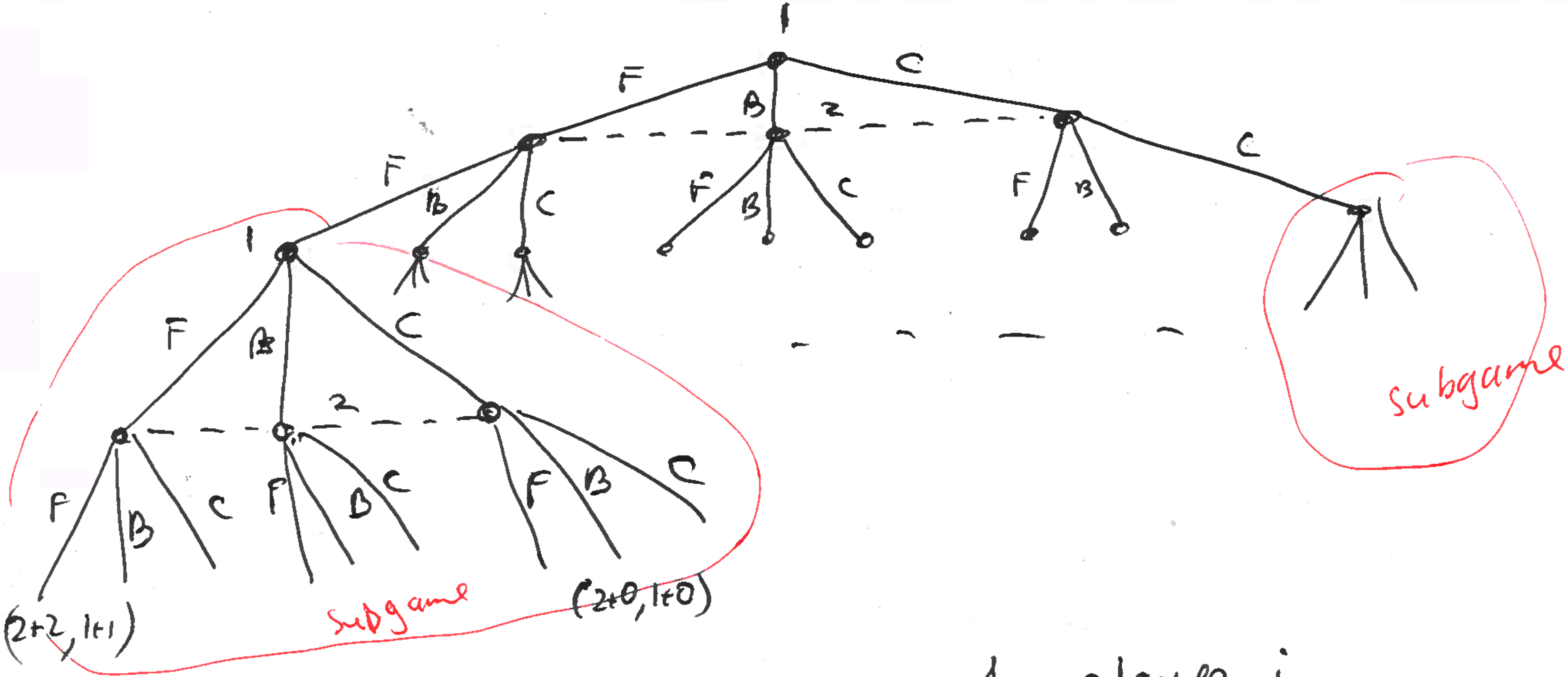
$G(2)$ = Game G played twice

→ play G first time

→ observe choices of the players

→ play G again

→ payoffs of $G(2)$ = sum of payoffs from the two plays.



A pure strategy s_i of player i specifies how to play in each round of the game depending on what happened before:

Round 1: $s_i^1 \in \{F, B, C\}$

Round 2: $s_i^2 : \{F, B, C\} \times \{F, B, C\} \to \{F, B, C\}$

strategy of player i

$$G = \begin{array}{c|c|c|c|} & F & B & C \\ \hline F & 2,\text{[illegible]} & 0,0 & 6,0 \\ \hline B & 0,0 & 1,2 & 0,0 \\ \hline C & 0,0 & 0,0 & 5,5 \\ \hline \end{array}$$

Instructions Play C, then play F.

Round 2: (F, F) NE

Round 1: (C, C) → Player 1 has a profitable deviation to F

⇒ This is not SPE!

Instructions

For each player $i = 1, 2$:

$$s_i^1 = C$$

$$s_i^2 = \begin{cases} F, & \text{if } a_1 = C \\ B, & \text{if } a_1 \neq C \end{cases}$$

Player 1 Follow instructions → (C,C) then (F,F) ~ 5+2 = 7

Deviate in Round 2: → F is BR to F, → B is BR to B } no reason to deviate.

Deviate in Round 1: → to F: (F, C) then (B, B) ~ 6+1 = 7

↳ no reason to deviate.

⇒ Player 1 playing BR to Player 2's strategy.

Similarly, Player 2 playing BR to player 1's strategy.

⇒ SPE!

Repeated Game $G(T, \delta)$

- G ~ stage game
- "Stages" are rounds of play, $t = 1, 2, \dots, T$
- T ~ number of rounds, T can be ∞.
- $a^s = (a_1^s, a_2^s, \dots, a_N^s)$ the play in round s
- "history" $h^t = (a^1, a_1^2, \dots, a^{t-1})$ is a list of action profiles (or plays) in each round $s = 1, 2, \dots, t-1$

→ h^t is common knowledge at the start of round t (perfect recall)

→ H^t = set of possible histories h^t

→ $H^1 = \{h^1\}$ is the null history

→ H^{T+1} or H^∞ is the set of complete histories of play (terminal nodes)

A <u>behavior strategy</u> of player i is $\sigma_i = (\sigma_i^1, \sigma_i^2, \dots, \sigma_i^T)$ where $\sigma_i^t : H^t \to \Delta(A_i)$

After observing history h^t, player i chooses action $a_i \in A_i$ with probability $\sigma_i(h_t)[a_i]$

Payoffs

Discount factor $\delta \in [0,1]$

→ today's value of getting one unit of payoff in the next round.

→ Probability that the play continues to the next round.

Assume

- δ is the same for all players
- δ is common knowledge
- We may have $\delta = 1$ or $T = \infty$ but not both.

Payoffs $$U_i(h^{T+1}) = \sum_{t=1}^{T} \delta^{t-1} u_i(a^t) = u_i(a^1) + \delta u_i(a^2) + \delta^2 u_i(a^3) + \dots + \delta^{T-1} u_i(a^T).$$

Same payoff x in every round:

$$U_i(h^{T+1}) = \sum_{i=1}^{T} \delta^{t-1} \cdot x = \begin{cases} \frac{1-\delta^T}{1-\delta} x & \text{if } \delta < 1,\ T < \infty \\ T \cdot x & \text{if } \delta = 1,\ T < \infty \\ \frac{1}{1-\delta} x & \text{if } \delta < 1,\ T = \infty \end{cases}$$

Normalized payoffs

$$\hat{u}_i(h^{T+1}) = \begin{cases} \frac{1-\delta}{1-\delta^T} \sum_{t=1}^{T} \delta^{t-1} u_i(a^t), & \delta<1,\ T<\infty \\ \frac{1}{T} \sum \text{ - - - }, & \delta=1,\ T<\infty \\ \cancel{\frac{1}{1-\delta}}\,(1-\delta) \sum \text{ - - - }, & \delta<1,\ T=\infty. \end{cases}$$

$\hat{u}_i(s) = \hat{u}_i(s_1, \dots s_N)$ ~ the expected (normalised) payoff of player i when the players play strategies $(s_1, s_2, \dots, s_N)$.

Principle of Optimality (One-shot deviation principle)

A strategy profile $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_N)$ is a SPE of game $G(T, \delta)$ if and only if at every history (on or off the equilibrium path) no player can increase her expected (normalized) utility by deviating from σ_i exactly once (at that history) and playing σ_i again afterwards.

Finitely Repeated Games, $T < \infty$

	C	D
C	-1, -1	-4, 0
D	0, -4	-3, -3

D strictly dominates C.

Consider $G(T, \delta)$

→ In round T, (D, D) must be played.

→ in round $T-1$, (D, D) must be played.

→ in round $T-2$, — , — —

...

⟹ Conclude: SPE $\boxed{\text{in all rounds, } (D,D) \text{ must be played, regardless of what happened before}}$

Proposition In any repeated game (finite or infinite) playing a NE of the stage game in every round is SPE.

Proposition Suppose that stage game G has a unique NE, α^*, and let $T < \infty$. Then the only SPE of $G(T, \delta)$ is to play α^* in every round (after every history).

Infinitely Repeated Games, $T = \infty$

Let α_{-i} be a mixed action profile of all players other than i is the stage game.

Let $w_i(\alpha_{-i}) = \max_{a_i} u_i(a_i, \alpha_{-i})$ = player i's best response payoff to α_{-i}

Let $\underline{v}_i = \min_{\alpha_{-i}} w_i(\alpha_{-i}) = \min_{\alpha_{-i}} \left[\max_{a_i} u_i(a_i, \alpha_{-i}) \right]$

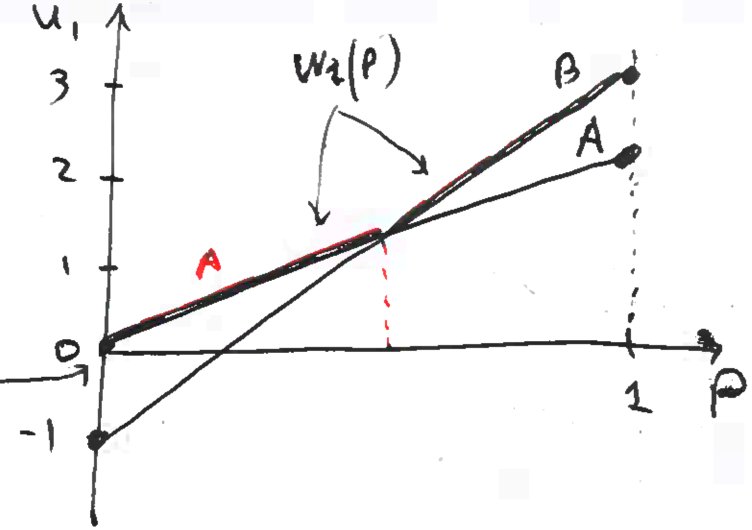
~ the minmax payoff of player i.

The <u>minmax strategy</u> used against i is the action profile m_i that delivers the minmax payoff to i: $m_i = \arg\min_{\alpha_{-i}} w_i(\alpha_{-i})$

<u>Proposition</u> In a repeated game, the normalized payoff of player i in any SPE is at least $\underline{v}_i$

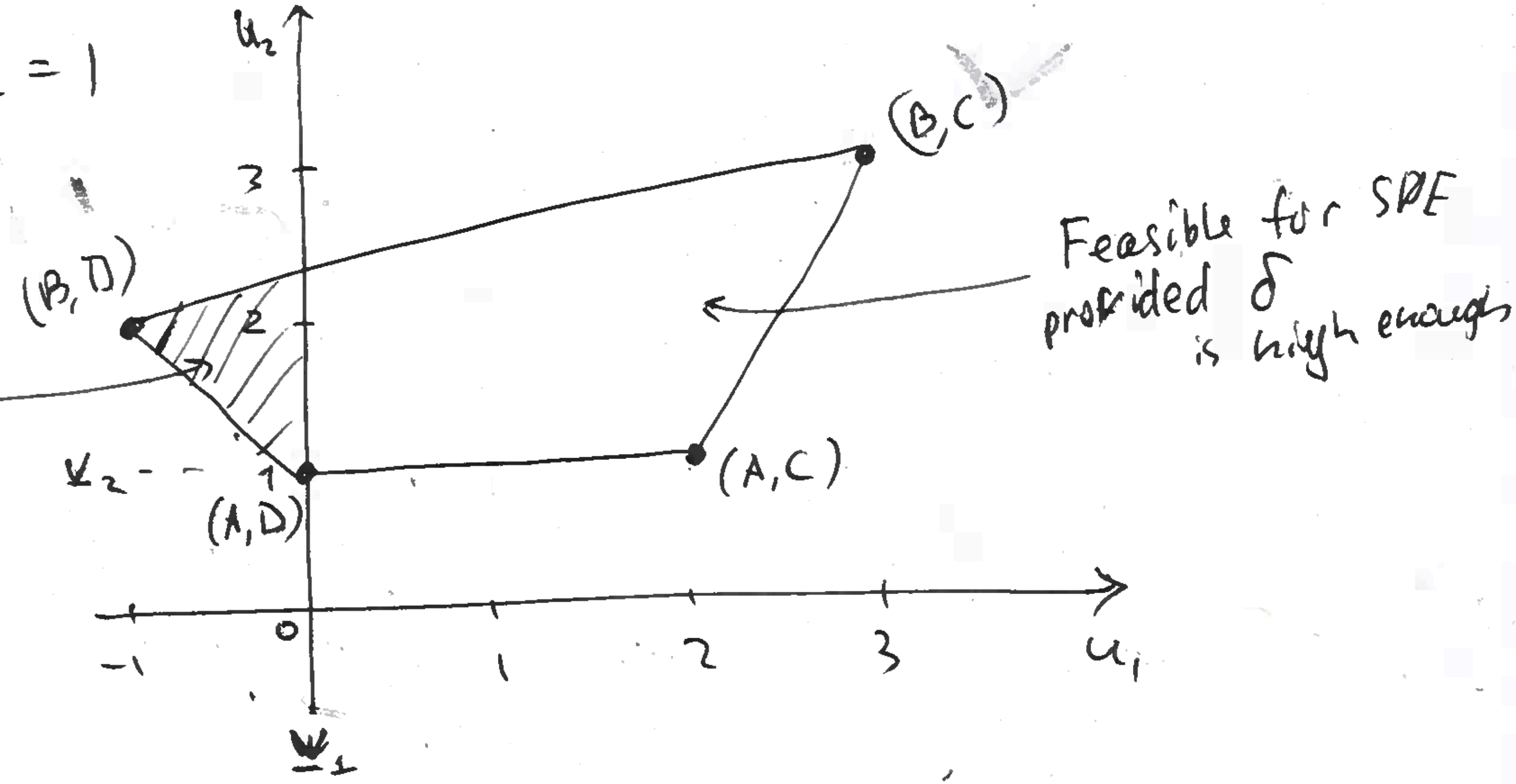
Example

	C	D
A	2, 1	0, 1
B	3, 3	−1, 2
	p	$1-p$



$\underline{v}_1 = \min\limits_{p} w_1(p) = 0$

Similarly, $\underline{v}_2 = 1$



Folk Theorem

Let V be the set of feasible payoffs of G

$$V = \text{Convex Hull}\{(u_1(a), u_2(a), \ldots, u_n(a)) : a \in A\}.$$

Proposition (Fudenberg & Maskin) Let $T = \infty$

Choose any vector of payoffs $v \in V$ such that

$$v_i > \underline{v}_i \quad \text{for each player } i.$$

Then there exists a discount factor threshold $\underline{\delta} \in (0,1)$ such that for every $\delta \in (\underline{\delta}, 1)$ there exists a SPE σ^* of $G(\infty, \delta)$ with $\hat{u}_i(\sigma^*) = v_i$ for each player i.