Repeated Games

Example

$G =$

	F	B	C
F	$^*2, 1^*$	$0, 0$	$^*6, 0$
B	$0, 0$	$^*1, 2^*$	$0, 0$
C	$0, 0$	$0, 0$	$5, 5^*$

NE: (F, F)

(B, B)

$$\Big(\underbrace{\left(\tfrac{2}{3}, \tfrac{1}{3}, 0\right)}_{\substack{F \quad B \quad C \\ \text{Pl. 1}}}, \underbrace{\left(\tfrac{1}{3}, \tfrac{2}{3}, 0\right)}_{\substack{F \quad B \quad C \\ \text{Pl. 2}}}\Big)$$

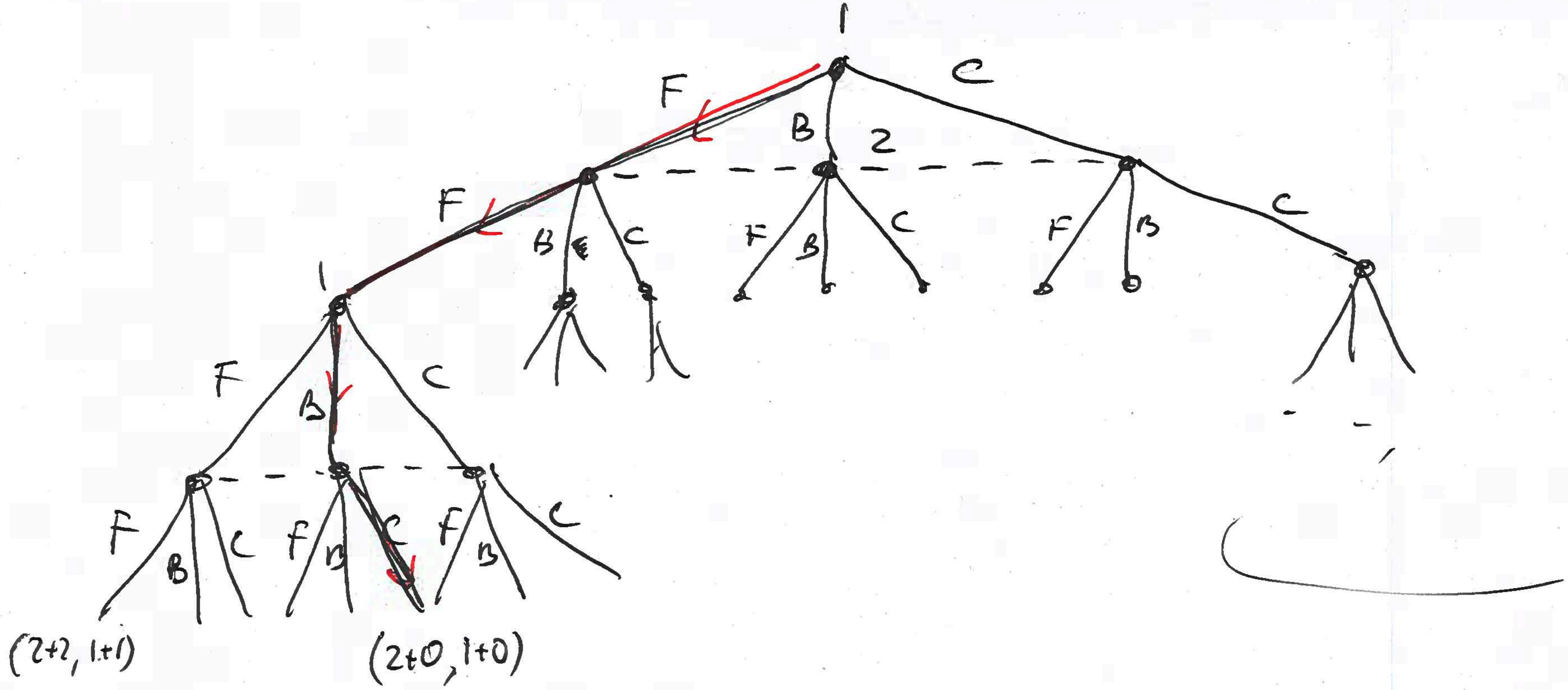
$G(2)$ = Game G is played twice

→ play G first time

→ observe choices of the players

→ play G again

→ payoffs of $G(2)$ = sum of payoffs from the two plays.



A pure strategy s_i of player i specifies how to play in each game depending on what happened before.

Game: $s_i^1 \in \{F, B, C\}$

$s_i^2 : \{F, B, C\} \times \{F, B, C\} \to \{F, B, C\}$

$$G = \begin{array}{c|c|c|c|} & F & B & C \\ \hline F & 2,1 & 0,0 & 6,0 \\ \hline B & 0,0 & 1,2 & 0,0 \\ \hline C & 0,0 & 0,0 & 5,5 \\ \hline \end{array}$$

Consider the following strategies for each player $i = 1,2$:

$$\boxed{s_i^1 = C \qquad s_i^2 = \begin{cases} F, & \text{if } a_1 = C \\ B, & \text{if } a_1 \neq C \end{cases}} \quad \text{for each } i=1,2$$

Is this SPE?

Player 1

Follow instructions $\to$ (C,C) then (F,F) ~ $5+2=7$

- Deviate in 2nd round $\to$ F is BR to F, B is BR to B } no reason to deviate
- Deviate in 1st round to F $\to$ $(F,C) \to (B,B)$ ~ $6+1=7$

$\Rightarrow$ Player 1 has no profitable deviation

Player 2

Follow instructions $\to$ (C,C) then (F,F) ~ $5+1=6$

- Dev in 2nd round $\to$ no reason to deviate, similarly to player 1.
- Dev in 1st round $\to$ worse off.

$\Rightarrow$ Player 2 has no profitable deviation.

Repeated Game $G(T, \delta)$

- G ~ stage game
- "Stages" are rounds of play, $t = 1, 2, \ldots, T$
- $a^s = (a_1^s, a_2^s, \ldots, a_N^s)$ the play in round s
- "history" $h^t = (a^1, a^2, \ldots, a^{t-1})$ is a list of action profiles (or plays) in each round $s = 1, 2, \ldots, t-1$

  → h^t is commonly known at the start of stage t

  → H^t = set of possible histories h^t

  → $H^1 = \{h^1\}$ is the null history

  → H^{T+1} or H^∞ is the set of complete histories (plays) (terminal nodes)

A <u>behavior strategy</u> of player i is

$$\sigma_i = (\sigma_i^1, \sigma_i^2, \ldots, \sigma_i^T) \quad \text{where} \quad \sigma_i^t : H^t \to \Delta(A_i)$$

After observing history h^t, player i chooses action $a_i \in A_i$ with probability $\sigma_i(h^t)[a_i]$

Payoffs

Discount factor $\delta \in [0,1]$

↳ today's value of getting one unit of payoff in the next round

↳ Probability that the play continues to the next round

Assume

- δ is the same for all players
- δ is common knowledge.

Payoffs $U_i(h^{T+1}) = u_i(a^1) + \delta u_i(a^2) + \delta^2 u_i(a^3) + \ldots + \delta^{T-1} u_i(a^T)$

Same payoff x in every round

$$\rightarrow U_i(h^{T+1}) = x + \delta \cdot x + \delta^2 \cdot x + \ldots + \delta^{T-1} \cdot x = \begin{cases} \frac{1-\delta^T}{1-\delta} \cdot x, & \delta < 1 \\ T \cdot x, & \delta = 1 \end{cases}$$

Normalized Payoffs

$$\hat{U}_i(h^{T+1}) = \begin{cases} \frac{1-\delta}{1-\delta^T} U_i(h^{T+1}) & \text{if } \delta < 1 \\ \frac{1}{T} U_i(h^{T+1}) & \text{if } \delta = 1 \end{cases}$$

For $T = \infty$ (Assume $\delta < 1$)

$$\hat{U}_i(h^{\infty}) = (1-\delta) U_i(h^{\infty}) = (1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} u_i(a^t).$$

$\hat{U}_i(s) = \hat{U}_i(s_1, \ldots, s_N)$ ~ the expected (normalised) payoff of player i from strategy profile s.

Principle of Optimality (One-shot deviation principle)

A strategy profile σ is a SPE of game $G(T, \delta)$ if and only if at every history (on or off equilibrium path) no player i can increase her expected (normalised) utility by deviating from σ_i exactly once (at that history) and playing σ_i again afterwards.

Finitely Repeated Games $T < \infty$

	C	D
C	-1, -1	-4, 0
D	0, -4	-3, -3

D strictly dominates C

Consider $G(T, \delta)$.

→ in stage T, (D, D) must be played.

→ in stage $T-1$, (D, D) must be played

→ in stage $T-2$, (D, D) must be played

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⇒ Conclude: in all stages (D, D) must be played.

<u>Proposition</u> In any repeated game (finite or infinite) playing a NE of the stage game in every stage is SPE.

<u>Proposition</u> Suppose that stage game G has a unique NE, α^*, and let $T < \infty$. Then <u>the only SPE</u> of $G(T, \delta)$ is to play α^* in every stage.

Infinitely Repeated Games $T = \infty$

Let α_{-i} be a mixed action profile of all players other than i in stage game

Let $w_i(\alpha_{-i}) = \max_{a_i} u_i(a_i, \alpha_{-i})$ = player i's best response payoff to α_{-i}

Let $\underline{v}_i = \min_{\alpha_{-i}} W_i(\alpha_{-i}) = \min_{\alpha_{-i}} \left[\max_{a_i} u_i(a_i, \alpha_{-i})\right]$

~ the minmax payoff of player i

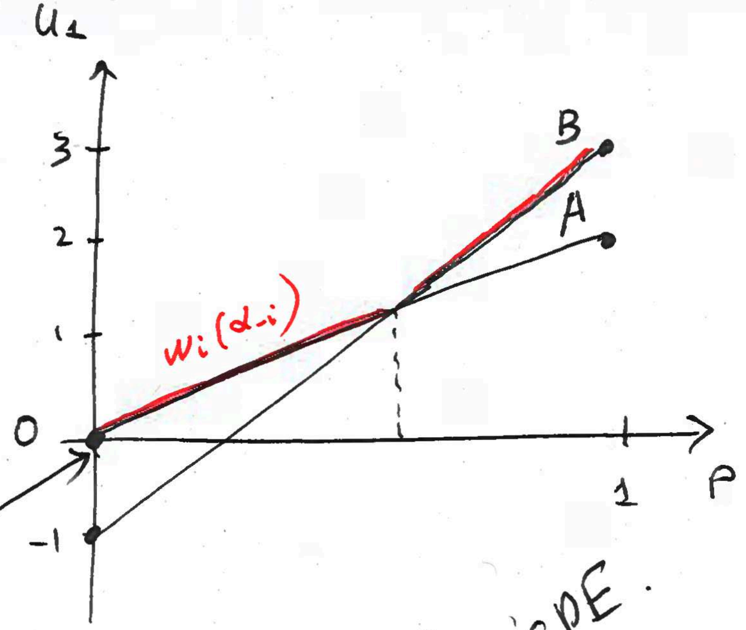
The minmax strategy used against player i is the action m_{*i} that delivers the minmax payoff.

$$m_{*i} = \arg\min_{\alpha_{-i}} W_i(\alpha_{-i})$$

Proposition In any repeated game, the payoff of player i in any SPE is at least $\underline{v}_i$.

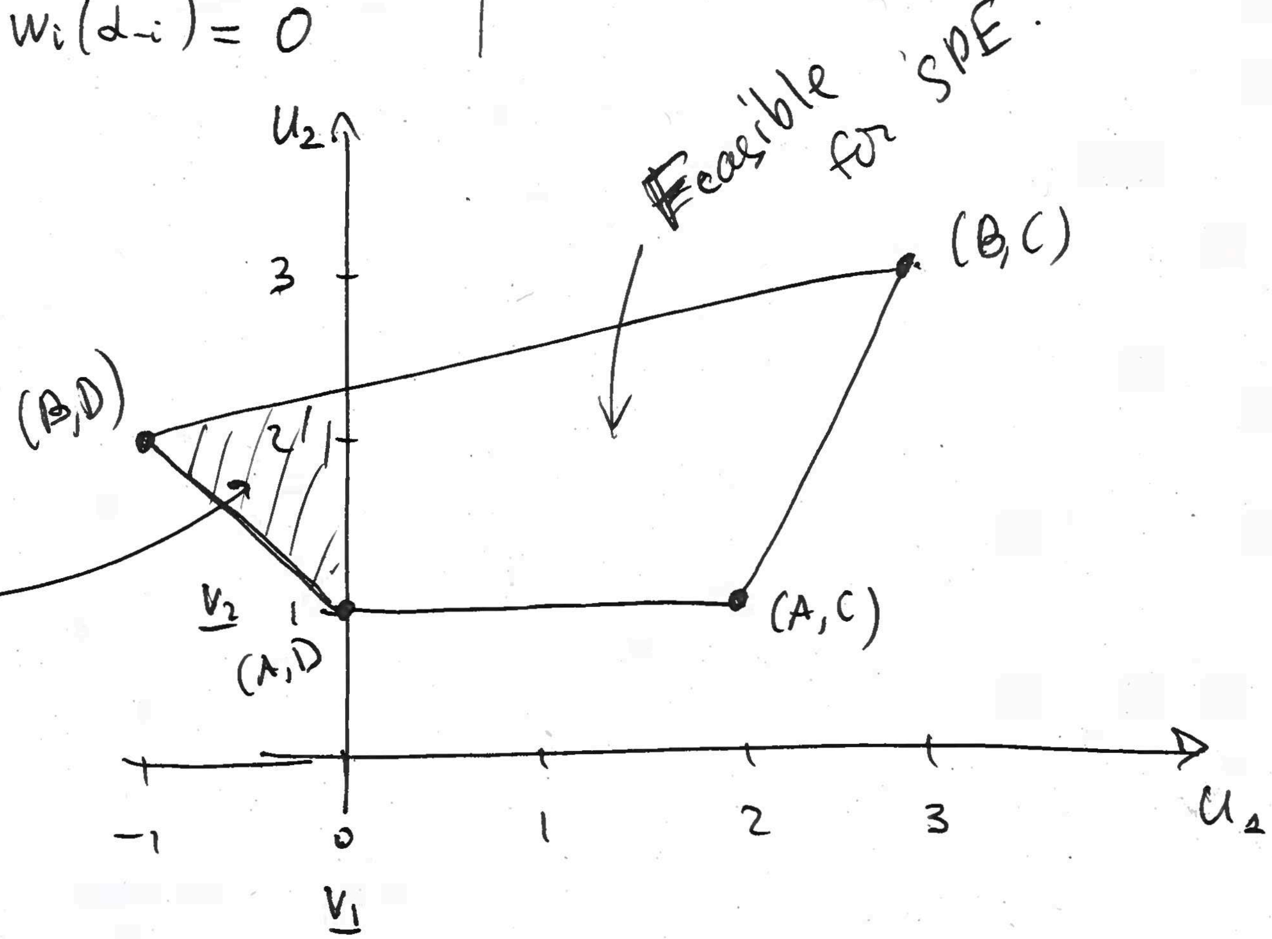
<u>Example</u>

	C	D
A	2, 1	0, 1
B	3, 3	−1, 2
	p	1 − p



$\underline{V}_1 = \cancel{\min_{\alpha_1 \in \{A,B\}}} \min_{p} w_i(\alpha_{-i}) = 0$

Similarly, $\underline{V}_2 = 1$



Folk Theorem

Homework

Folk Theorem

Let V be the set of feasible payoffs of G

$$V = \text{Convex Hull}\left\{ (u_1(a), u_2(a), \dots, u_N(a)) : a \in A \right\}$$

Proposition (Fudenberg & Maskin) Let $T = \infty$.

Choose any vector of payoffs $v \in V$ such that

$$v_i > \underline{v_i} \quad \text{for each player } i$$

Then there exists a ~~discount~~ discount factor threshold $\underline{\delta} \in (0,1)$ such that for every discount factor $\delta \in (\underline{\delta}, 1)$ there exists a SPE σ^* of $G(\infty, \delta)$ with $\hat{U}_i(\sigma^*) = v_i$ for each player i.