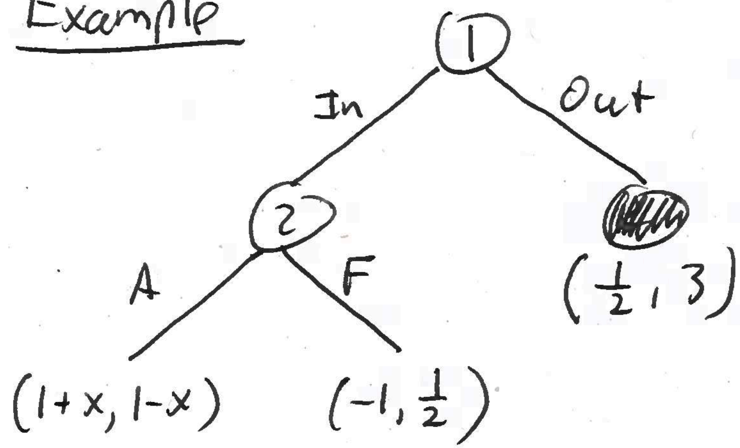
Games of Incomplete Information

Players have uncertainty about:

- → payoffs
- → game structure
- → info of other players
- → rationality of others
- → players

Example



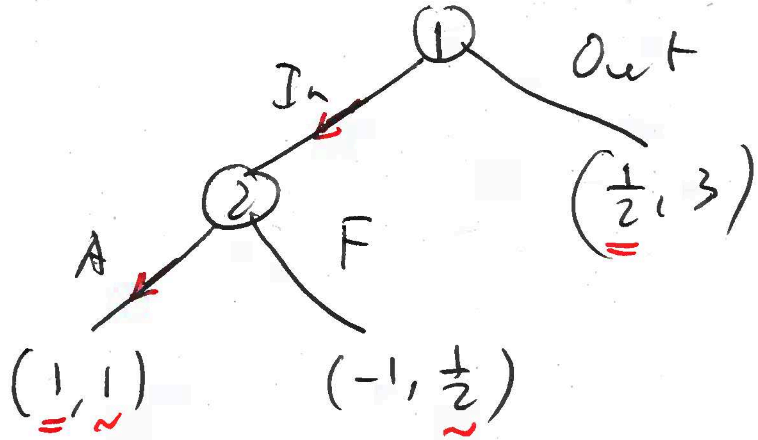
$x \begin{cases} \nearrow 1 \sim \text{prob } 1/2 \\ \searrow -1 \sim \text{prob } 1/2 \end{cases}$

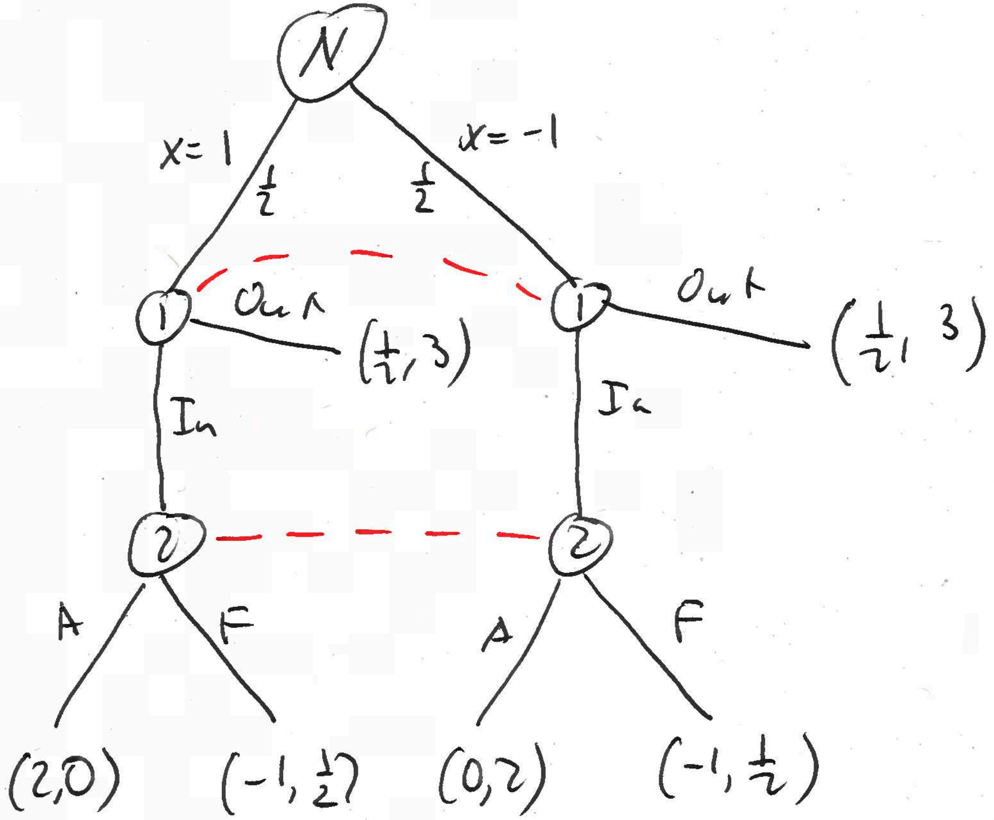
x unknown to both

→ "Symmetric Information";
no player has informational
advantage.

$x \rightarrow \frac{1}{2}\cdot(1) + \frac{1}{2}\cdot(-1)$

Solution → Replace
uncertain payoffs
by expected payoffs



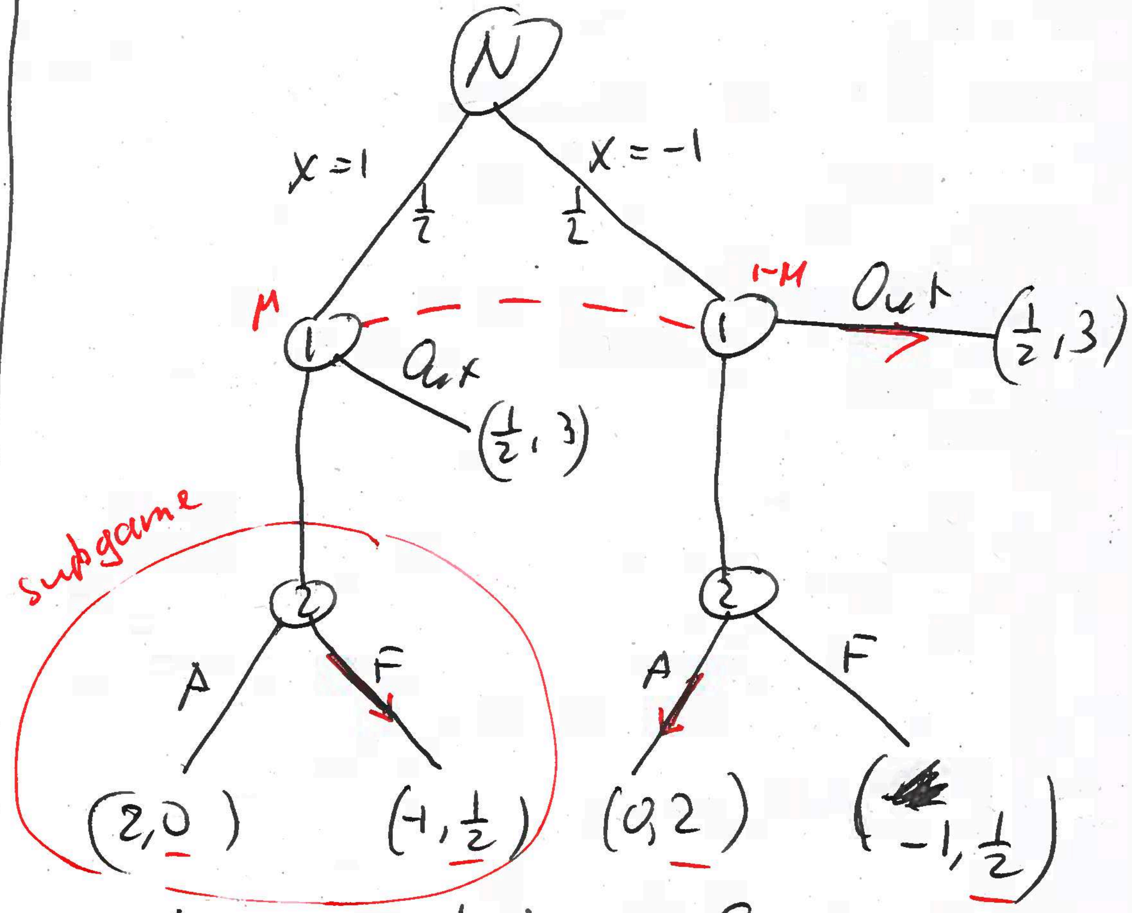


Symmetric Info

Equilibrium

(Out, FA)

$\mu = \frac{1}{2}$



Asymmetric Info:

Player 1 does not know x

But player 2 does.

Pl 1: Out $\rightarrow \frac{1}{2}$

In $\rightarrow \mu(-1) + (1-\mu)\cdot(0) < \frac{1}{2}$.

Let x be a decision node in Player i's info. set h.
Then $\mu_i(x|h)$ is the probability of being in node x conditional on being in info. set h.

belief

$$1 \geq \mu_i(x|h) \geq 0, \quad \sum_{x \in h} \mu_i(x|h) = 1.$$

Def An assessment is a behavior strategy profile σ and beliefs μ where $\mu : X \to [0,1]$, where $\mu(\cdot|h)$ specifies beliefs at each info. set h.

Assessment = choices + beliefs.

Def An assessment (σ, μ) is sequentially rational if playing σ_i maximizes the expected utility of player i given that player's beliefs at each info. set where that player moves, for each player i.

$\Rightarrow \sigma_i$ is a best response of i given μ.

__Def.__ Beliefs μ are __consistent__ if they satisfy Bayes' rule whenever "possible" (that is, for every info. set that is reached with positive probability)

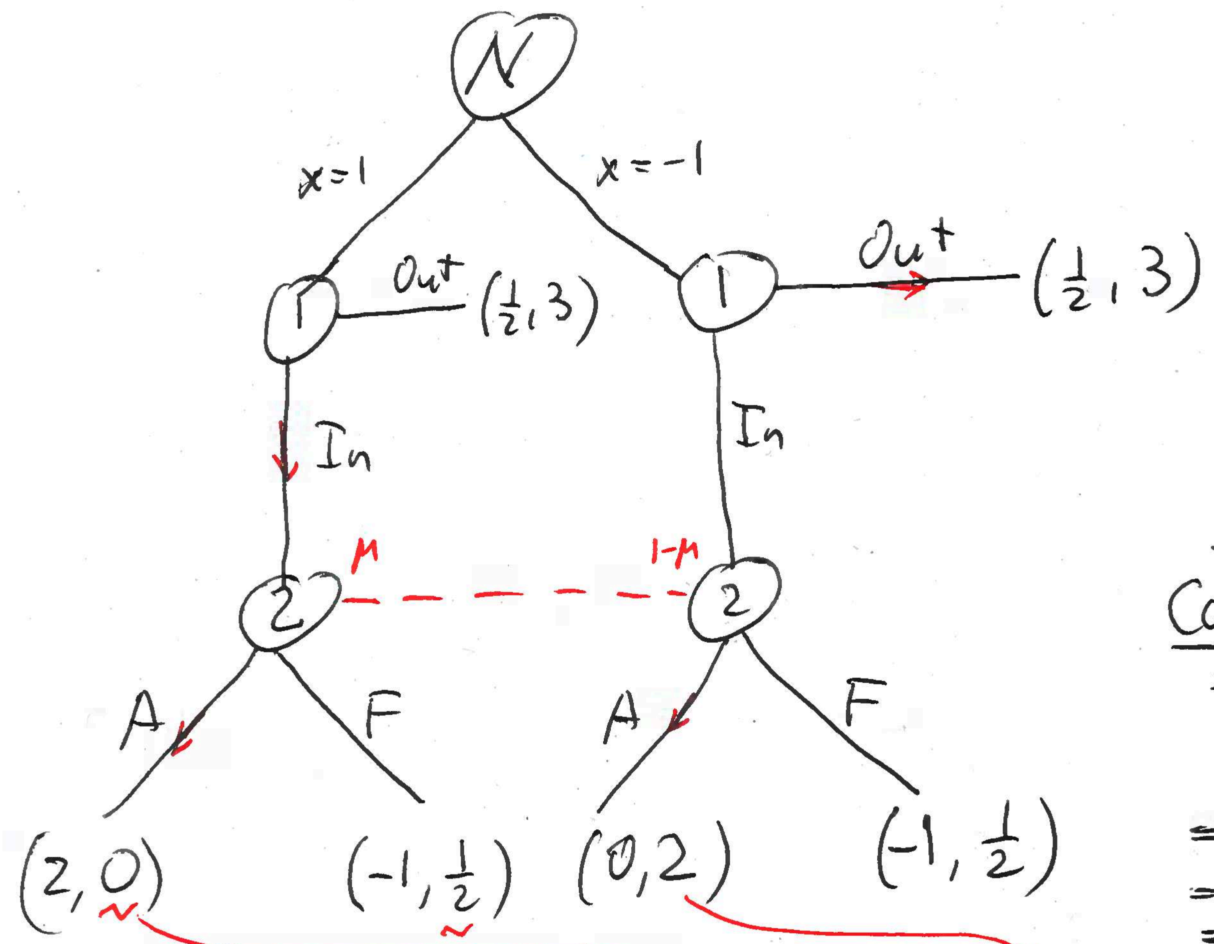
consistent = correspond to what really happens

__Def__ An assessment (σ, μ) is a __perfect Bayesian equilibrium__ (PBE) if

(a) σ is sequentially rational given μ,

(b) μ is consistent under σ.

Asymmetric Info: Player 1 knows x but player 2 does not.



Case 1. $\sigma_2 = A$

P1: $\begin{cases} Out \to \frac{1}{2}, & In \to 2 \\ Out \to \frac{1}{2}, & In \to 0 \end{cases}$

(In, Out)

$\Rightarrow$ Consistent belief $\mu = 1$

$\Rightarrow$ A not BR for Player 2!

$\Rightarrow$ Not PBE

Case 2 $\sigma_2 = F$

$\Rightarrow$ P1 $\quad Out \to \frac{1}{2}, \quad In \to -1$

$\Rightarrow (Out, Out)$

$\Rightarrow$ Every belief $\mu \in [0,1]$ is consistent

$\Rightarrow$ Assume $\mu \geq 3/4$

$\Rightarrow \sigma_2 = F$ is BR

$$BR_2(\mu) = \begin{cases} F, & \mu > \frac{3}{4} \\ \text{Any action}, & \mu = \frac{3}{4} \\ A, & \mu < \frac{3}{4} \end{cases}$$

Player 2

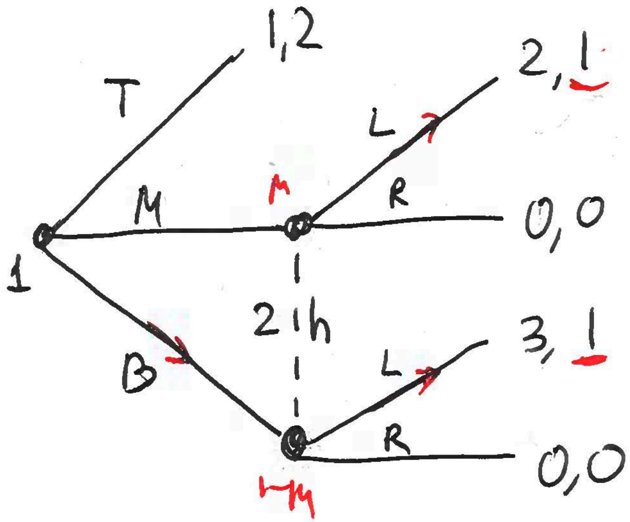
$$A \longrightarrow \mu \cdot (0) + (1-\mu) \cdot (2) = 2 - 2\mu$$

$$F \longrightarrow \mu \cdot \left(\tfrac{1}{2}\right) + (1-\mu) \cdot \left(\tfrac{1}{2}\right) = \frac{1}{2}$$

$2 - 2\mu = \frac{1}{2}, \quad 4 - 4\mu = 1, \quad 4\mu = 3, \quad \mu = \frac{3}{4}$

$$\boxed{\text{PBE: } \left((Out, Out), \left(F, \mu \geq \tfrac{3}{4}\right)\right)}$$

Example: Complete Info.



	L	R
T	$1, 2^*$	$^*1, 2^*$
M	$2, 1^*$	$0, 0$
B	$^*3, 1^*$	$0, 0$

NE = SPE

(T, R) ~ problem

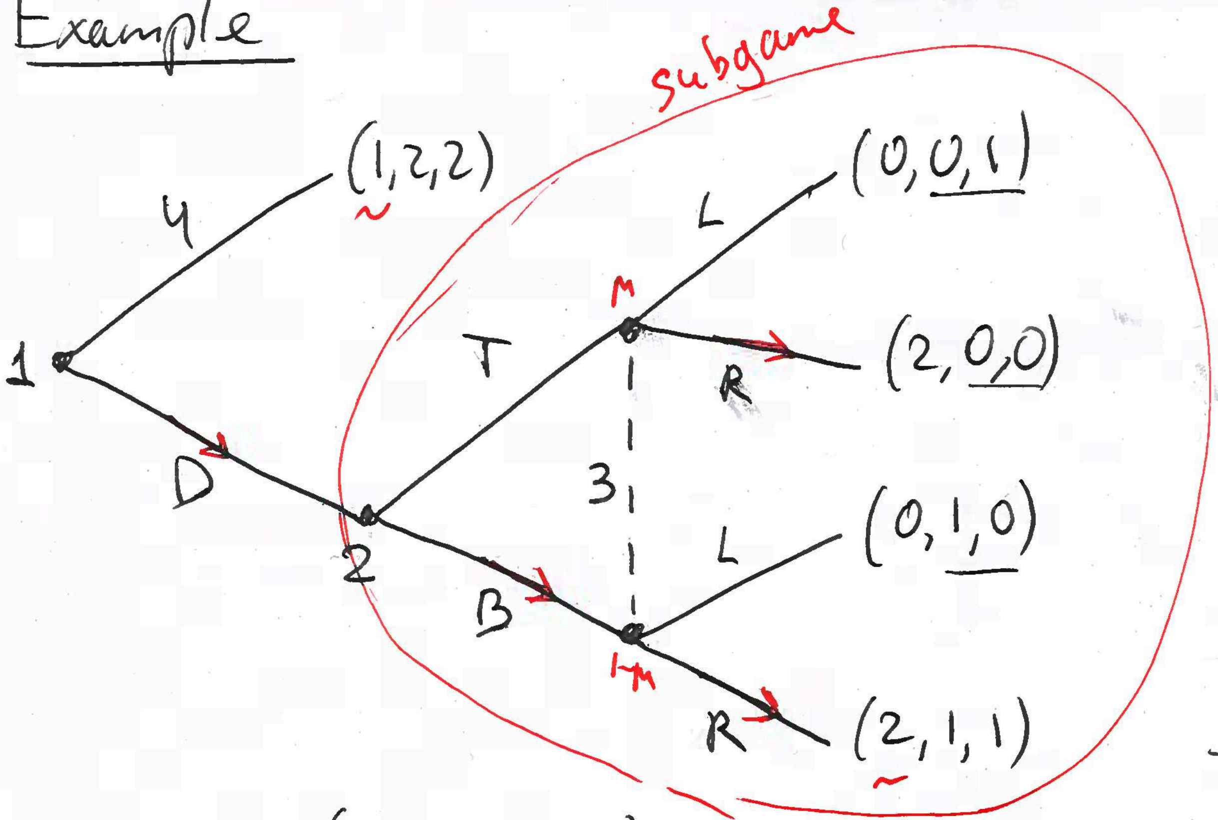
$$L \to \mu \cdot (1) + (1-\mu) \cdot (1) = 1$$
$$R \to \mu \cdot (0) + (1-\mu) \cdot (0) = 0$$
$\Rightarrow$ Choose L

$\Rightarrow$ T $\to$ 1, M $\to$ 2, B $\to$ 3 $\Rightarrow$ Choose B

PBE: $(B, (L, \mu = 0))$

$\Rightarrow \quad \mu = 0.$

Example



pl. 3

pl.2	L	R
T	0, 1*	0, 0
B	*1, 0	*1, 1*

(B,R) is NE

SPE = PBE

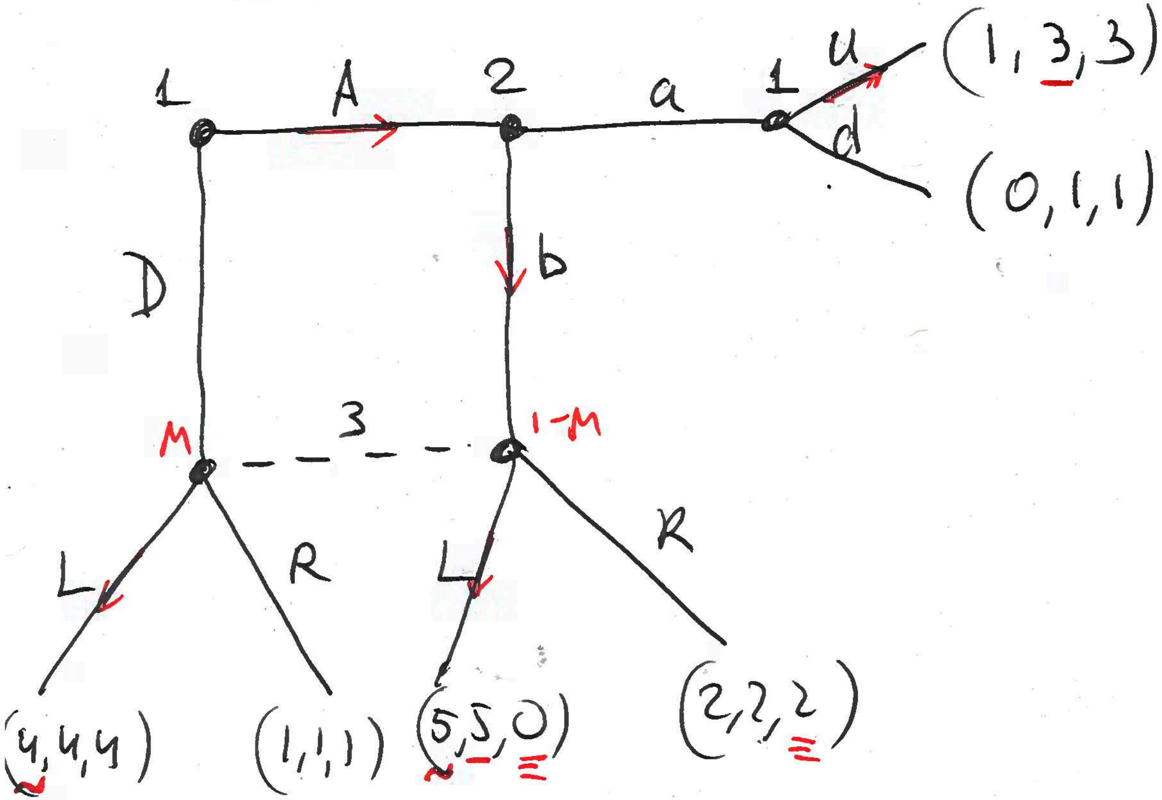
$SPE = (D, B, R)$

$PBE = (D, B, (R, \mu = 0))$.

Proposition If (σ, μ) is a PBE, then σ is an SPE.

$PBE \subseteq SPE \subseteq NE$

Example: Selten's Horse



Case 1 $\sigma_3 = L$

$\Rightarrow \sigma_2 = b$

$\Rightarrow \sigma_1 = A$

$\Rightarrow \mu = 0$

$\Rightarrow \sigma_3 = L$ not BR!

Case 2 $\sigma_3 = R$

$\Rightarrow \sigma_2 = a$

$\Rightarrow \sigma_1$ indifferent between D and A

$\rightarrow$ Assume D with pos prob

$\Rightarrow \mu = 1$

$\Rightarrow \sigma_3 = R$ is not BR!

$\rightarrow$ Assume $\sigma_1 = A$

$\Rightarrow \mu$ is any

$\left(\mu \leq \frac{2}{3}\right)$

Suppose $\mu = 0$

$\Rightarrow \sigma_3 = R$ BR of Player 3.

PBE: $(Au, a, (R, \mu = 0))$.