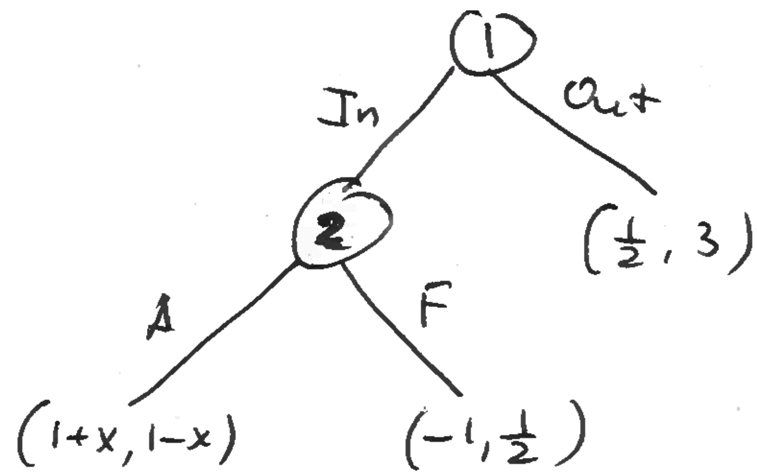
# Games of Incomplete Information

Players have uncertainty about:

→ payoffs
→ game structure
→ info of other players
→ rationality of others
→ players

Example



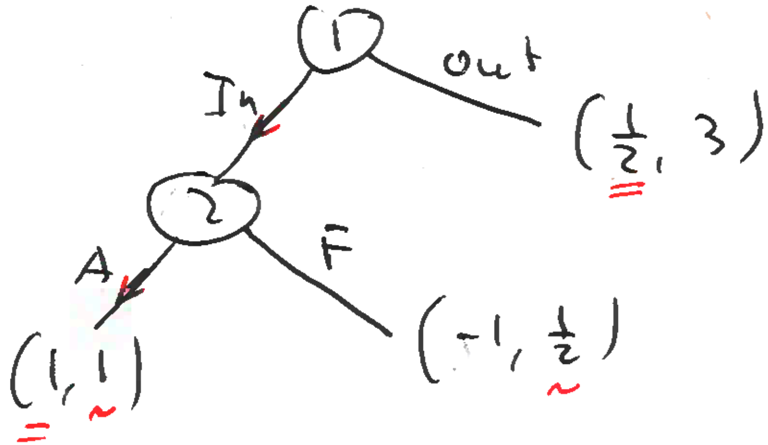

$x \to 1$ ~prob 1/2
$x \to -1$ ~prob 1/2

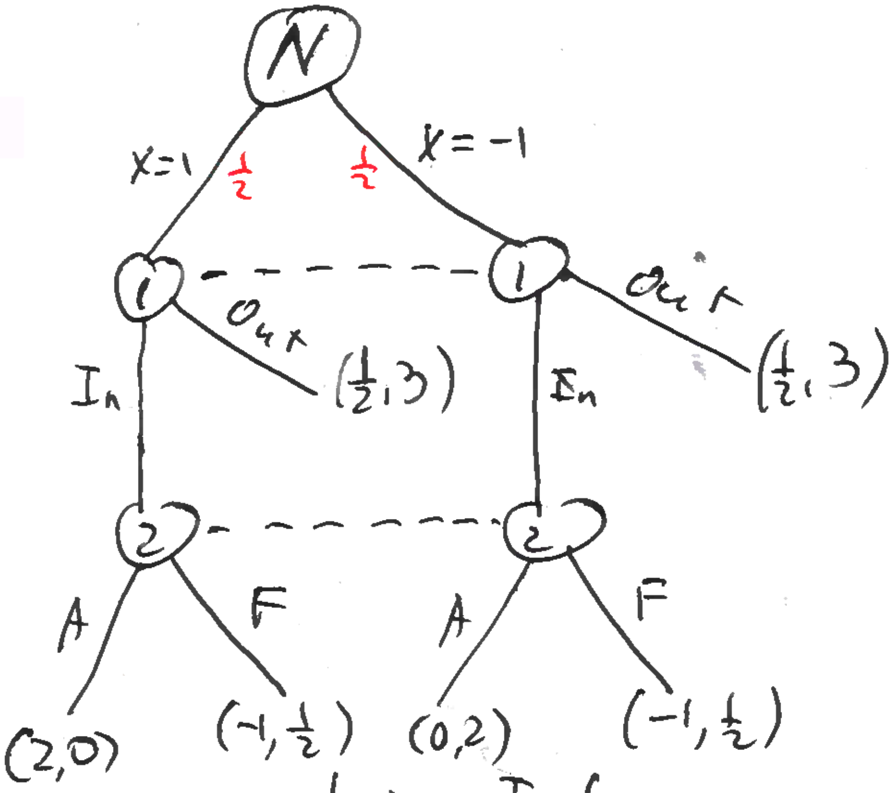
Case 1: x unknown to both players.

→ "Symmetric information"
no player has an informational advantage.

Solution → Replace uncertain payoffs by their expected payoffs.

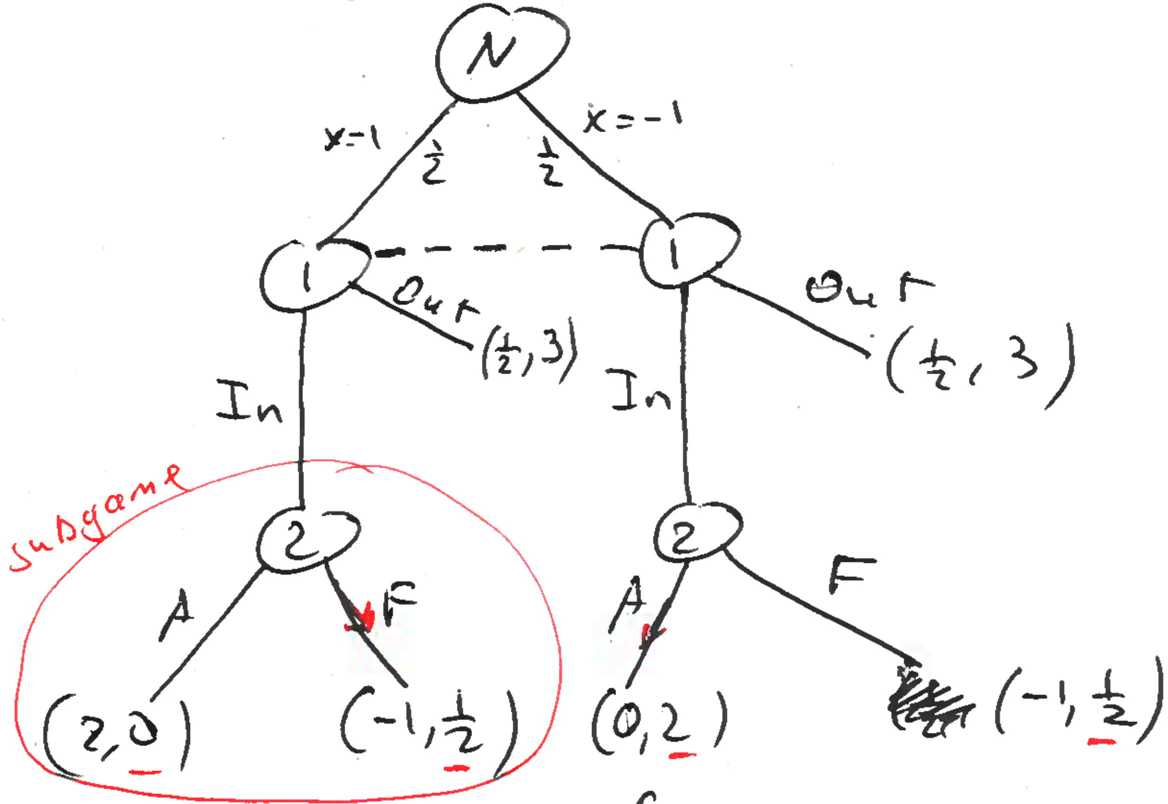
$(1+x, 1-x) \xrightarrow{\text{replaced}} (1, 1)$



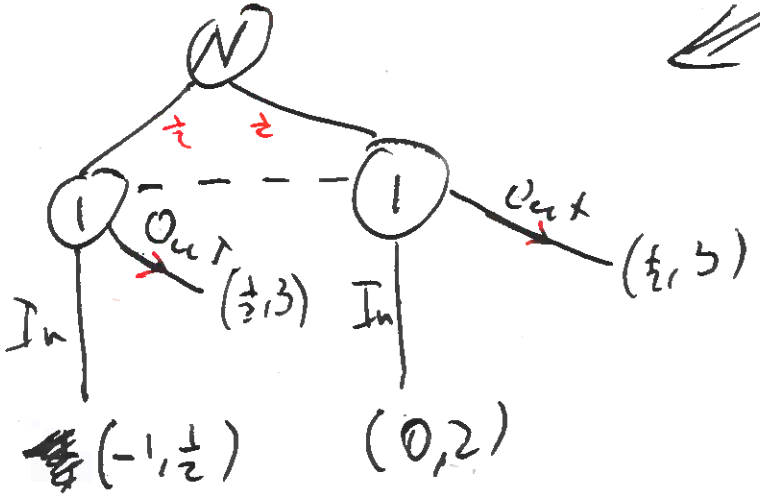



Symmetric Info

Case 2: Player 1 does not observe x, but player 2 does.




Asymmetric Info




Out $\rightarrow \frac{1}{2}$

In $\rightarrow (-1)\cdot\frac{1}{2} + 0\cdot\frac{1}{2} = -\frac{1}{2}$

Equilibrium: (Out, FA)

Let $x$ be decision node in player $i$'s info. set $h$.
Then $\underline{\mu_i(x|h)}$ is the probability of being in node $x$ conditional on being in info. set $h$.

belief

$$0 \le \mu_i(x|h) \le 1, \quad \sum_{x \in h} \mu_i(x|h) = 1$$

Def. An <u>assessment</u> is a behavior strategy profile $\sigma$ and beliefs $\mu$ where $\mu : X \to [0,1]$, and $\mu(\cdot|h)$ specifies beliefs at each info. set $h$.

Assessment = strategies + beliefs

Def. An assessment $(\sigma, \mu)$ is <u>sequentially rational</u> if playing $\sigma_i$ maximizes the expected utility of player $i$, given that player's beliefs, at each info. set where that player moves for each player $i$.

$\Rightarrow$ $\sigma_i$ is a best response of $i$ given $\mu$.

Def. Beliefs $\mu$ are consistent under $\sigma$ if they satisfy Bayes' rule whenever possible. (that is, for every info. set that is reached with positive probability).

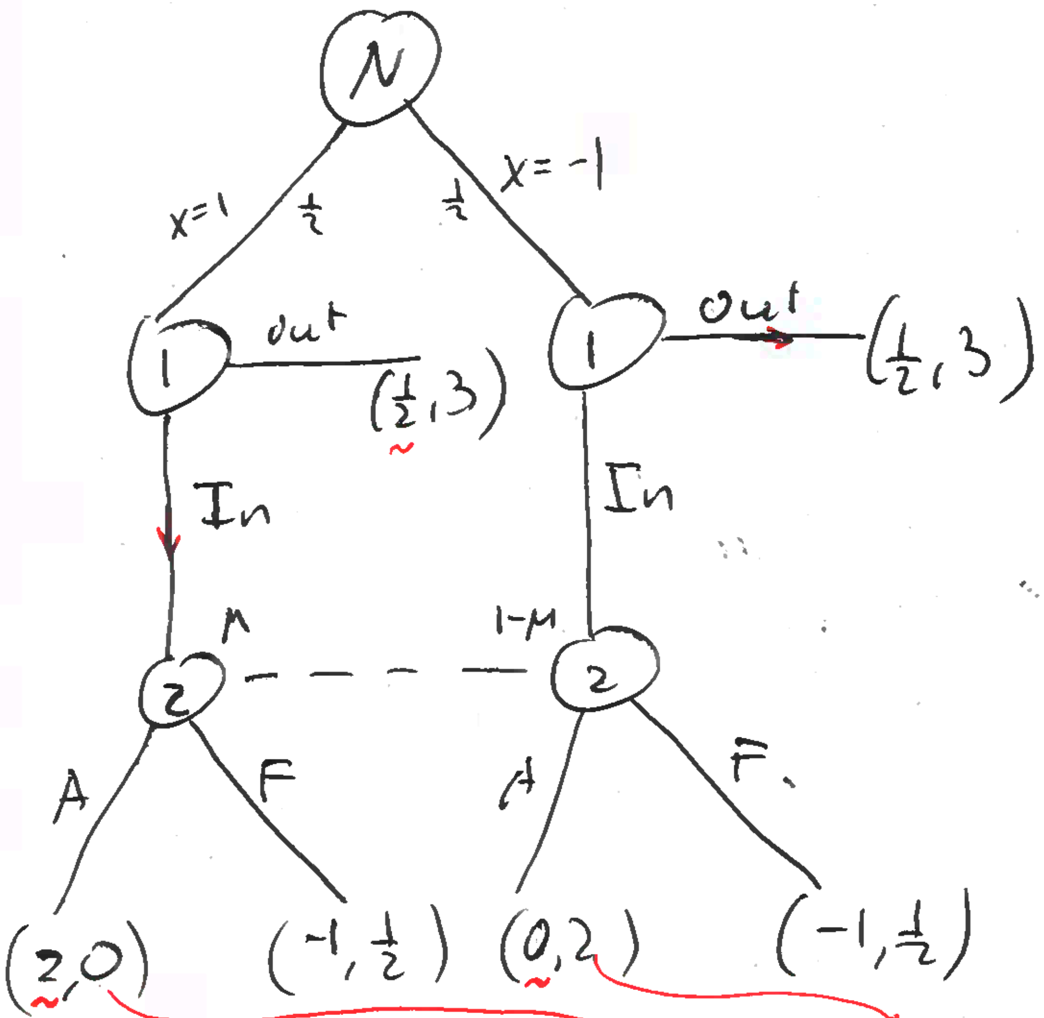
consistent = correspond to what really happens.

Def. An assessment $(\sigma, \mu)$ is a perfect Bayesian equilibrium (PBE) if

(a) $(\sigma, \mu)$ is sequentically rational

(b) $\mu$ is consistent under $\sigma$.

## Asymmetric info: Player 1 knows x but Player 2 does not.




**Player 2:** $A \rightarrow \mu \cdot 0 + (1-\mu)\cdot 2 = 2-2\mu$

$F \rightarrow \mu \cdot \frac{1}{2} + (1-\mu)\cdot \frac{1}{2} = \frac{1}{2}$

$2-2\mu = \frac{1}{2} \Rightarrow \mu = \frac{3}{4}$

$$BR_2(\mu) = \begin{cases} F, & \mu > \frac{3}{4} \\ \text{Any action}, & \text{if } \mu = \frac{3}{4} \\ A, & \text{if } \mu < \frac{3}{4} \end{cases}$$

**Case 1** $\sigma_2 = A$

**Player 1:** $\begin{cases} Out \rightarrow \frac{1}{2}, In \rightarrow 2 \\ Out \rightarrow \frac{1}{2}, In \rightarrow 0 \end{cases}$

(In, Out)

$\Rightarrow$ Consistent belief. $\mu = 1$

$\Rightarrow$ A is not BR for player 2!

$\Rightarrow$ Not PBE

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**Case 2** $\sigma_2 = F$

**Player 1:** $\begin{cases} Out \rightarrow \frac{1}{2} & In \rightarrow -1 \\ Out \rightarrow \frac{1}{2} & In \rightarrow -1 \end{cases}$

(Out, Out)

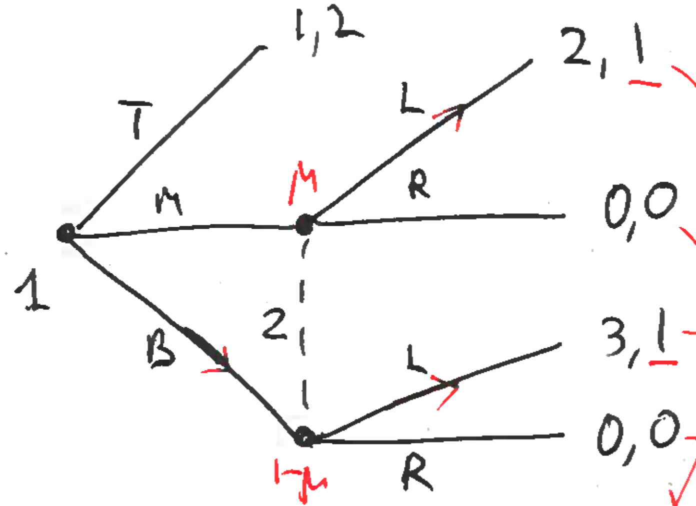
$\Rightarrow$ Every belief $\mu \in [0,1]$ is consistent

$\Rightarrow$ Assume $\mu \geq \frac{3}{4}$

$\Rightarrow \sigma_2 = F$ is BR for player 2 $\Rightarrow$ PBE

PBE: $((Out, Out), (F, \mu \geq \frac{3}{4}))$

# Example




| | L | R |
|---|---|---|
| T | $1,2^*$ | $^*1,2^*$ |
| M | $2,1^*$ | $0,0$ |
| B | $^*3,1^*$ | $0,0$ |

NE = SPE

But R is strictly dominated by L!

**Player 2**

$$L \to \mu\cdot(1) + (1-\mu)(1) = 1$$
$$R \to \mu\cdot(0) + (1-\mu)(0) = 0$$
$$BR_2(\mu) = L \quad \text{for all } \mu \in [0,1].$$

**Player 1**

$$\left.\begin{array}{l} T \to 1 \\ M \to 2 \\ B \to 3 \end{array}\right\} \Rightarrow \text{Choose } B.$$

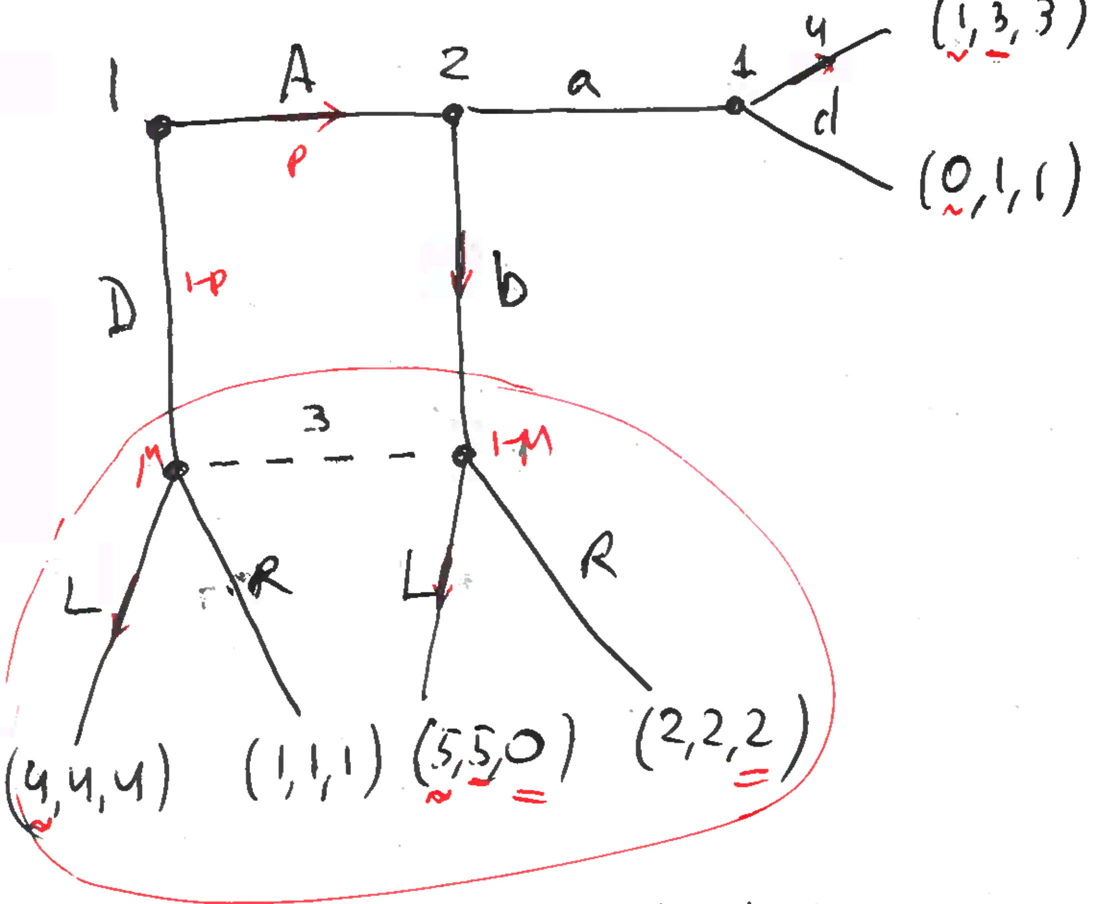
$$\Rightarrow \quad \mu = 0$$

PBE: $(B, (L, \mu=0))$

**Proposition** If $(\sigma, \mu)$ is a PBE, then $\sigma$ is an SPE.

$$PBE \subseteq SPE \subseteq NE$$

## Example: Selten's Horse




Player 3: $L \to \mu \cdot 4 + (1-\mu) \cdot 0 = 4\mu$

$R \to \mu \cdot 1 + (1-\mu) \cdot 2 = 2-\mu$

$$BR_3(\mu) = \begin{cases} L, & \mu > \frac{2}{5} \\ \text{Any}, & \mu = \frac{2}{5} \\ R, & \mu < \frac{2}{5} \end{cases}$$

PBE: $\left(Au, a, \left(R, \mu \leq \frac{2}{5}\right)\right)$.

Case 1 $\quad \sigma_3 = L$

$\Rightarrow \sigma_2 = b$

$\Rightarrow \sigma_1 = Au$

$\Rightarrow \mu = 0$

$\Rightarrow \sigma_3 = L$ not BR!

$\to$ not PBE.

Case 2 $\quad \sigma_3 = R$

$\Rightarrow \sigma_2 = a$

$\Rightarrow$ Player 1 indifferent between D and A

$\to$ Case 2.1: Player 1 $\to$ D

$\Rightarrow \mu = 1$

$\Rightarrow \sigma_3 = R$ is not BR!

$\Rightarrow$ not PBE

$\to$ Case 2.2: Player 1 $\to$ A.

$\Rightarrow \mu$ is any,

$\Rightarrow \sigma_3 = R$ is BR when $\mu \leq \frac{2}{5}$