# Game Theory

## Example

Player 1

~~1/2~~ ~~1/2~~ Rain God

| | Sun | Rain |
|---|---|---|
| No umbrella | 5 | 0 |
| Umbrella | 1 | 3 |

payoffs of player 1

beliefs

- structural uncertainty (Nature, Laws of physics)
- strategic — " — (endogenous, man-made)

Normal-form (strategic form, simultaneous-move, matrix-form) game consists of 3 elements:

1) Set of players $\{1, \dots, N\}$

2) Set of actions $A_i$ for each player $i$

$A = A_1 \times A_2 \times \dots \times A_N \to$ set of action profiles

$a = (a_1, a_2, \dots, a_N) \in A$ ~ action profile (or a play)

3) Payoffs: ~~[illegible]~~ $u_i : A \to \mathbb{R}$ for each $i = 1, \dots, N$

For today, action = strategy

# Dominance

Action $a$ **strictly dominates** action $a'$

if $u(a,b) > u(a',b)$ for all $b$.

Action $a$ **weakly dominates** action $a'$

if 1) $u(a,b) \geq u(a',b)$ for all $b$

2) There exists $b'$ s.t $u(a,b') > u(a',b')$.

Action $a$ is **dominant** if it strictly dominates all other actions

— " — " **weakly dominant** if it weakly dominates all other actions.

Action $a$ is **undominated** (admissible) if it is not weakly dominated by any other action.

## Examples

| player 1 \ Player 2 | C | D |
|---|---|---|
| C | -1 | -4 |
| D | 0 | -3 |

$$\underset{\uparrow\ P1\ \ \uparrow\ P2}{u_1(D,C)} = 0 > u_1(C,C) = -1$$

$$u_1(D,D) = -3 > u_1(C,D) = -4$$

$\Rightarrow$ D strictly dominates C for Player 1.

D is undominated

D is dominant.

| Pl$_1$ \ Pl.2 | C | D |
|---|---|---|
| C | 0 | -4 |
| D | 0 | -3 |

$\Rightarrow$ D weakly dominates C

D is undominated

D is weakly dominant.

**Observations** A rational player

→ will never play a strictly dominated action

→ will always play the dominant action (if there is one)

→ might play a weakly dominated action.

# Belief rationality and Best Response

**Beliefs** Player 1's beliefs about Player 2's actions are given by a probability distribution

$$p = [\, p(b_1), p(b_2), \dots, p(b_K) \,]$$

where $p(b_k)$ is the probability that player 2 plays action $b_k$, $k = 1, 2, \dots, K$.

Expected payoff of Player 1 from playing action $a$ given belief $p$ is

$$u(a, p) = \sum_{k=1}^{K} p(b_k)\, u(a, b_k)$$

If Player 2 plays action $b_k$ with prob. 1,

so $p(b_k) = 1$, $p(b_j) = 0$ for all $j \neq k$,

Then Player 1's exp. payoff $u(a, b_k)$.

**Observation** A rational player will always choose an action that maximizes $u(a, p)$ given beliefs $p$.

**Def** Action $a$ is a **best response to beliefs $p$** if $u(a,p) \geq u(a',p)$ for every $a' \in A_1$

**Def** The **best-response correspondence** $BR(p)$ is the set of best-response actions to belief $p$,

$$BR(p) = \underset{a \in A_1}{\operatorname{argmax}}\ u(a,p)$$

# Sun-Rain Game

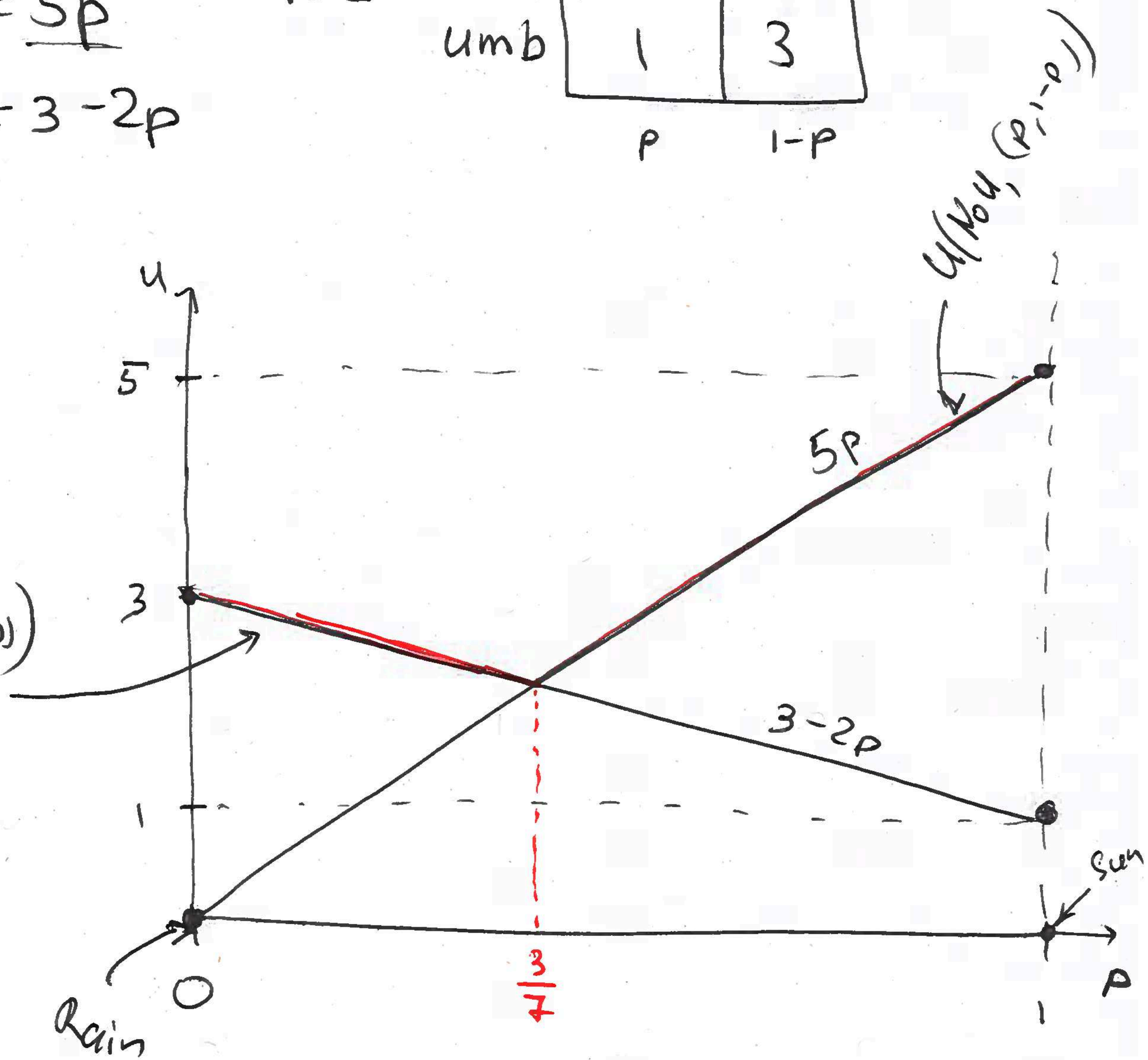
Pl.1

| | Sun | Rain |
|---|---|---|
| No umb | 5 | 0 |
| umb | 1 | 3 |
| | $p$ | $1-p$ |

$$u(NoU, \overset{\text{Sun}}{(p}, \overset{\text{Rain}}{1-p)}) = 5 \cdot p + 0 \cdot (1-p) = \underline{5p}$$

$$u(Um, (p, 1-p)) = 1 \cdot p + 3 \cdot (1-p) = 3 - 2p$$

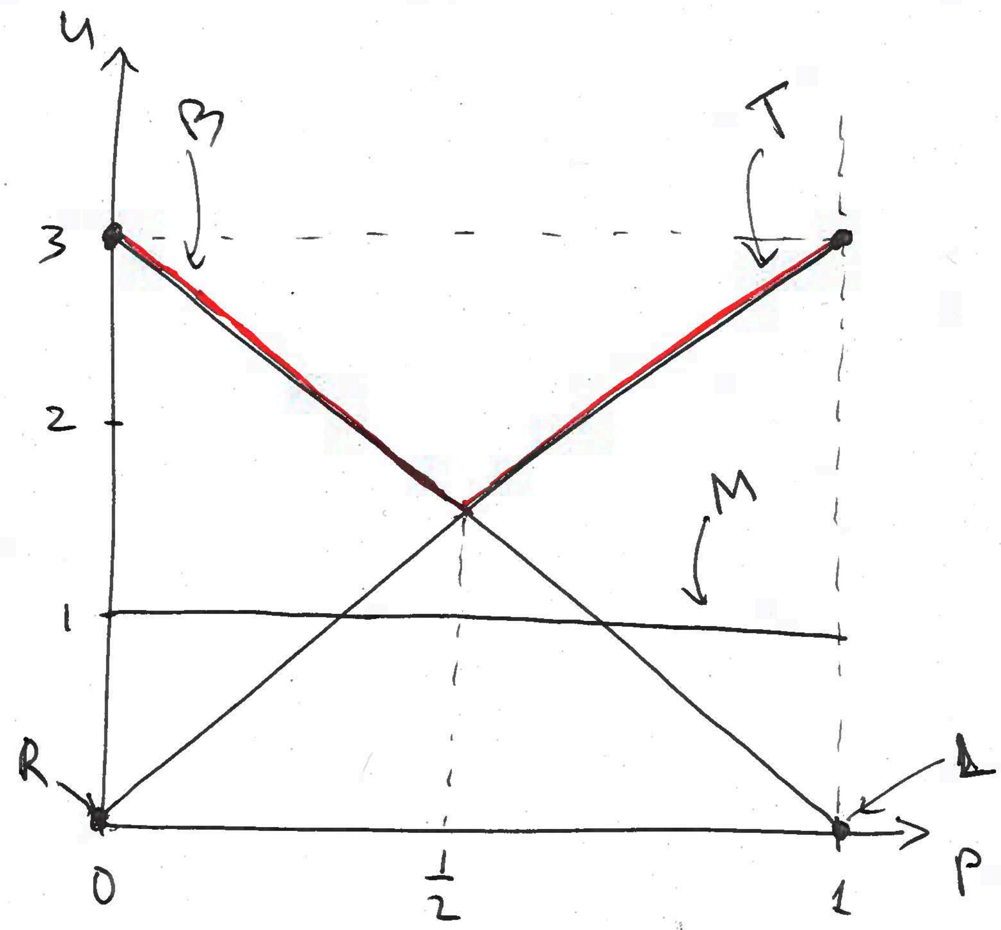
$$BR(p) = \begin{cases} Um, & \text{if } p < \frac{3}{7} \\ \{Um, NoU\}, & \text{if } p = \frac{3}{7} \\ NoU, & \text{if } p > \frac{3}{7} \end{cases}$$



## Example

Player 1

| | L | R |
|---|---|---|
| T | 3 | 0 |
| M | 1 | 1 |
| B | 0 | 3 |
| | $p$ | $1-p$ |




$$BR(p) = \begin{cases} B & \text{if } p < \frac{1}{2} \\ \{B, T\}, & \text{if } p = \frac{1}{2} \\ T, & \text{if } p > \frac{1}{2} \end{cases}$$

M is <u>never a best response</u>

**Def.** Action $a$ is <u>never a best response</u> (NBR) if it is not a best response to any belief $p \in \Delta(A_2)$, that is,

$$a \notin BR(p) \quad \text{for all } p \in \Delta(A_2)$$

→ A rational player will never choose a NBR action

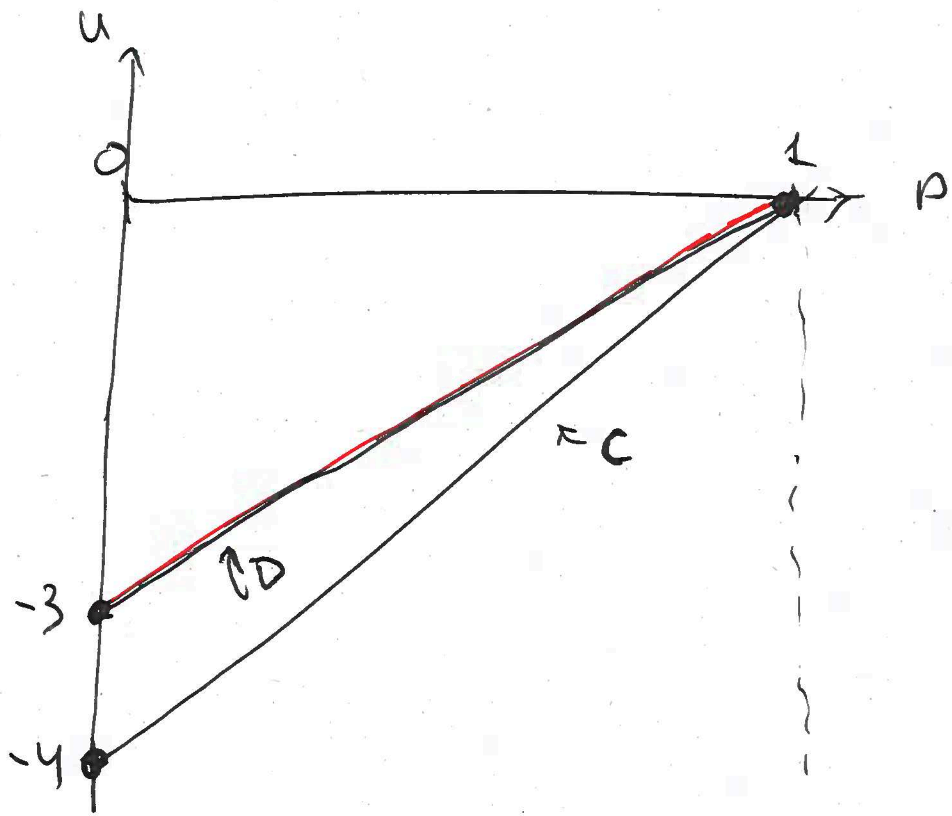
**Proposition** A strictly dominated action is NBR.

→ A weakly dominated action may or may not be a NBR.

→ A NBR action may or may not be dominated.

| | C | D |
|---|---|---|
| C | 0 | -4 |
| D | 0 | -3 |
| | $p$ | $1-p$ |

$$BR(p) = \begin{cases} D, & \text{if } p < 1 \\ \{D, C\}, & p = 1 \end{cases}$$




C is weakly dominated (by D)

but C is a best response when $p=1$