Expected Utility Theory: Doubts

Allais Paradox

$C_1 = 5M \qquad C_2 = 1M \qquad C_3 = 0$

$u(c)$ utility function $\quad u_1 = u(C_1) > u_2 = u(C_2) > u_3 = u(C_3)$

$$L_1 = (0, 1, 0) \quad \succ \quad L_2 = (0.10,\ 0.89,\ 0.01)$$

$$L_3 = (0,\ 0.11,\ 0.89) \quad \prec \quad L_4 = (0.10,\ 0,\ 0.90)$$

$$EU(L_1) = 0 \cdot u_1 + 1 \cdot u_2 + 0 \cdot u_3 = u_2 \quad > \quad EU(L_2) = 0.1 \cdot u_1 + 0.89 \cdot u_2 + 0.01 \cdot u_3$$

$$EU(L_3) = 0 \cdot u_1 + 0.11 \cdot u_2 + 0.89 \cdot u_3 \quad < \quad EU(L_4) = 0.1 \cdot u_1 + 0 \cdot u_2 + 0.9 \cdot u_3$$

$$+ (0.89 u_2 - 0.89 u_3) \qquad\qquad + (0.89 u_2 - 0.89 u_3)$$

$$u_2 \quad < \quad 0.1 u_1 + 0.89 u_2 + 0.01 u_3$$

→ Reference dependence

→ Different attitude towards gains and losses

Ellsberg Paradox

Urn 1	Urn 2
49 Red 51 Blue	Red ? Blue 100 in total

<u>Situation 1</u> Win 1000 if draw a Red, 0 if draw a Blue — majority chooses Urn 1

p = Prob. of Red in urn 2 ≡ <u>belief</u> about urn 2

<u>Situation 2</u> Win 1000 if draw a Blue, 0 — - - Red. — majority chooses Urn 1

→ Ambiguity aversion

Monetary Outcomes

x monetary outcome

$u(x)$ utility function (Bernoulli)

$f(x)$ density of x (probability density function, PDF)

Lottery

$(f_1, f_2, \dots, f_K; \alpha_1, \alpha_2, \dots, \alpha_K)$ ~ compound lottery

$$f(x) = \sum_{k=1}^{K} \alpha_k f_k(x) \quad \text{for all } x$$

Expected (monetary) value of lottery f: $\int x f(x) dx$

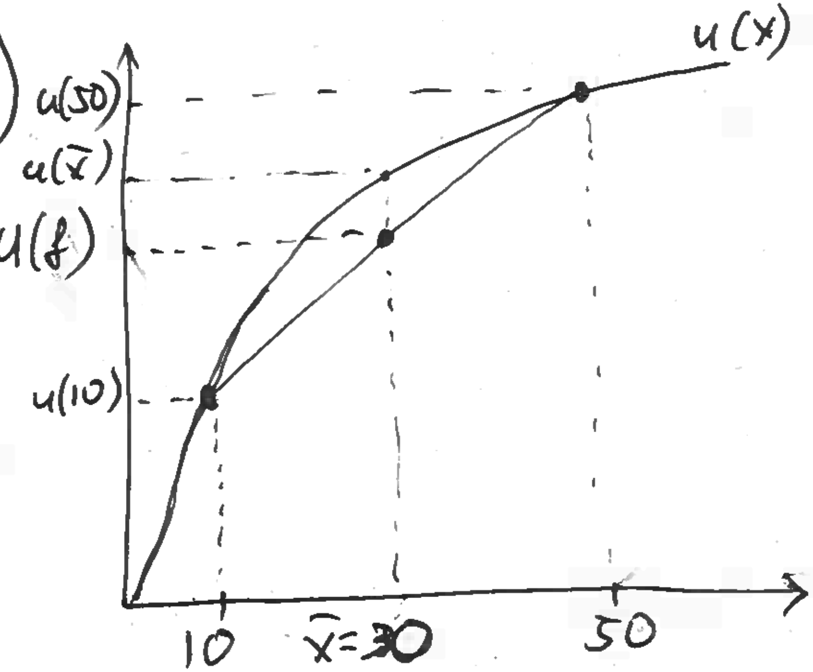
$$U(f) = \int u(x) f(x) dx$$

vN-M expected utility of f

Assume: u is increasing, continuous, bounded

Def A decision maker with a Bernoulli u.f. $u(x)$ is **risk averse** **(strictly)** if for every lottery f

$$\int u(x) f(x)\,dx \underset{(<)}{\leq} u\left(\int x f(x)\,dx \right)$$



Ex: $f \sim \left(10, 50; \frac{1}{2}, \frac{1}{2}\right)$

$$\bar{x} = 10 \cdot \tfrac{1}{2} + 50 \cdot \tfrac{1}{2} = 30$$

Expected monetary value

* **Risk-loving** (strictly) if

$$\int u(x) f(x)\,dx \underset{(>)}{\geq} u\left(\int x f(x)\,dx \right)$$

* Risk-neutral if

$$\int u(x) f(x)\,dx = u\left(\int x f(x)\,dx \right)$$

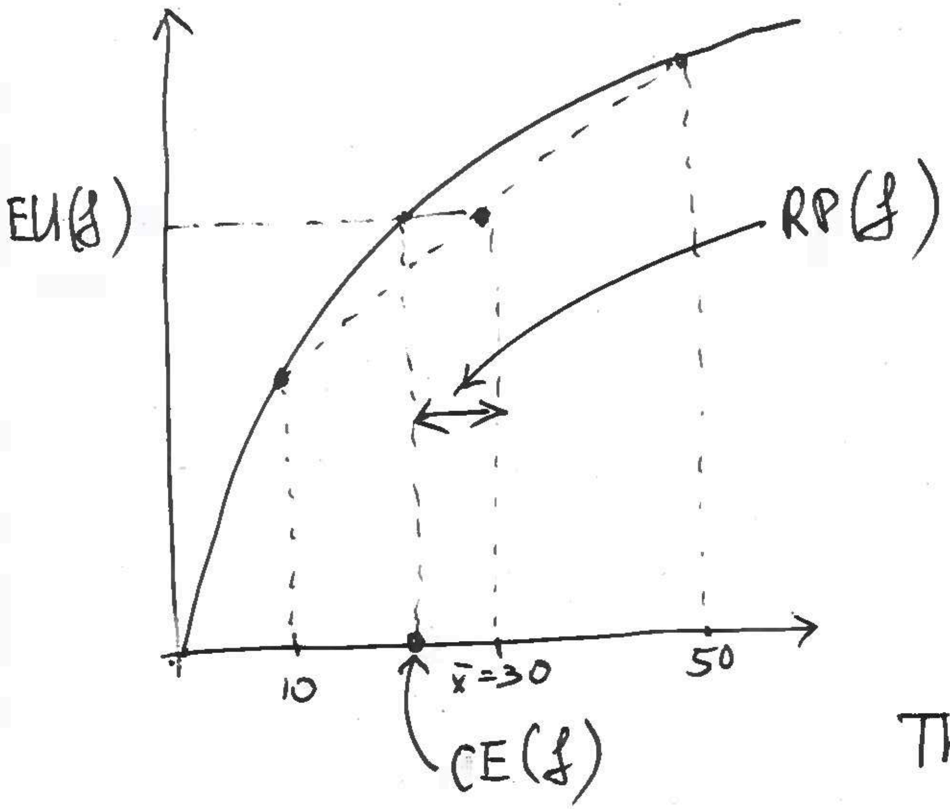
Def Given $u(\cdot)$, ~~an~~ we define the <u>certainty equivalent</u> of lottery f, denoted by $CE(f)$ as the amount of money for which the individual is indifferent between lottery f and the amount $CE(f)$:

$$\int u(x) f(x)\,dx = u(CE(f)).$$

Def The <u>risk premium</u> of lottery f denoted by $RP(f)$ is:

$$RP(f) = \underbrace{\int x f(x)\,dx}_{\text{expected value of } f} - \underbrace{CE(f)}_{\text{certainty equivalent of } f}$$

$RP(f)$ is negative if risk-loving.



Proposition

The following properties are equivalent:

(1) Decision maker is risk averse / risk loving / risk neutral

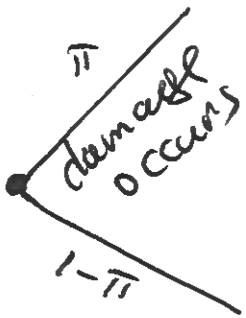
(2) $u(\cdot)$ is concave / convex / linear

(3) $CE(f) \leq / \geq / = \int x f(x) dx$ for all lotteries f,

(4) $RP(f) \geq / \leq / = 0$ for all f

Insurance

- Strictly risk averse decision maker
- W ~ initial wealth
- D ~ possible loss
- π ~ probability of loss
- q ~ price of insurance, per unit of cover
- α ~ cover (units of insurance bought)



$$W - D - \underbrace{\alpha \cdot q}_{\text{the premium}} + \underbrace{\alpha}_{\text{compensation (or cover)}}$$

$$W - \alpha \cdot q$$

$\alpha = 0$ ~ no insurance

$\alpha = D$ ~ full cover

How much cover α to buy?

<u>Assumptions</u> 1. Insurance is <u>actuarially fair</u> (the insurer makes zero profit)

$$\underbrace{\alpha q}_{\text{premium}} = \pi \cdot \alpha \qquad (*)$$

2. Full cover, $\alpha = D$, premium is affordable, $D \cdot q < W$

From $(*)$ $\Longrightarrow$ $\boxed{q = \pi}$

K-T conditions for maximization

$$\max_{\alpha} \quad (1-\pi)\, u(W-\alpha\pi) + \pi\, u(W-D-\alpha\pi+\alpha)$$

s.t. $\underline{\alpha \geq 0}$, $\quad \alpha\cdot\pi \leq W \iff \underline{W-\alpha\pi \geq 0}$

$$\mathcal{L} = (1-\pi)\, u(W-\alpha\pi) + \pi\, u(W-D-\alpha\pi+\alpha) + \mu\alpha + \lambda(W-\alpha\pi)$$

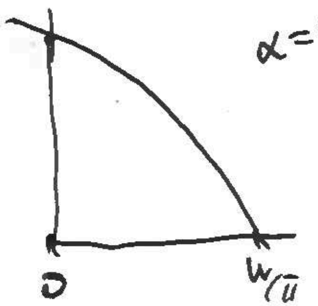
Lagrange multipliers

F.O.C. $$\frac{\partial\mathcal{L}}{\partial\alpha} = -\pi(1-\pi)u'(W-\alpha\pi) + \pi(1-\pi)u'(W-\alpha\pi-D+\alpha) + \mu - \lambda\pi = 0 \qquad (1)$$

$$\frac{\partial\mathcal{L}}{\partial\mu} = \alpha \geq 0 \qquad (2) \qquad\qquad \frac{\partial\mathcal{L}}{\partial\lambda} = W-\alpha\pi \geq 0 \qquad (3)$$

$$\mu \geq 0 \qquad (4) \qquad\qquad \lambda \geq 0 \qquad (6)$$

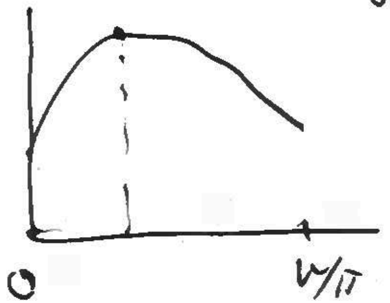
$$\alpha\cdot\mu = 0 \qquad (5) \qquad\qquad \lambda\cdot(W-\alpha\pi) = 0 \qquad (7)$$



$\alpha = 0$?

$\alpha = \frac{W}{\pi}$

?

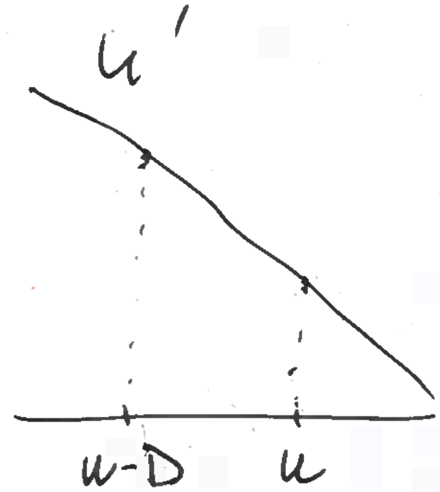


α interior ?

<u>Case 1</u> Suppose $\alpha^*=0$. $\Rightarrow$ $\lambda^*=0$ by (7)

By (1) we get

$$\underbrace{\pi(1-\pi)}_{>0}\Big(\underbrace{u'(w)-u'(w-D)}_{<0}\Big)=\underbrace{\mu^*}_{\geq 0 \text{ by } (4)}$$



Cannot be satisfied.

$\Rightarrow$ $\alpha^*\neq 0$.

<u>Case 2</u> Suppose $\alpha^*=w/\pi$ $\Rightarrow$ $\mu^*=0$

By (1) we get

$$\underbrace{-\pi(1-\pi)}_{<0}\Big(\underbrace{u'(0)-u'\big(\tfrac{w}{\pi}-D\big)}_{>0}\Big)=\underbrace{\lambda^*\pi}_{\geq 0 \text{ by } (6)}$$

Cannot be satisfied. $\frac{w}{\pi}-D>0$ by Assumption 2

$\Rightarrow$ $\alpha^*\neq w/\pi$

Case 3 Interior $\alpha \in (0, W/\pi)$

$$\Rightarrow \lambda^* = 0 \quad \mu^* = 0 \quad \text{by (5) and (7)}$$

$$(1) \Rightarrow u'(W - \alpha\pi) = u'(W - D - \alpha\pi + \alpha)$$

$$\Rightarrow W - \alpha\pi = W - D - \alpha\pi + \alpha$$

$$\Rightarrow \alpha^* = D$$

Optimal to buy full cover

$\alpha^* = D$

π: $W - D - \alpha^* \cdot \pi + \alpha^* = W - D \cdot \pi$

$1-\pi$: $W - \alpha^* \cdot \pi = W - D \cdot \pi$