Expected Utility Theory: Doubts

Allais Paradox $\qquad C_1 = 5M \qquad C_2 = 1M \qquad C_3 = 0$

$u(C)$ utility function $\qquad u_1 = u(C_1) \qquad u_2 = u(C_2) \qquad u_3 = u(C_3)$

$$L_1 = (0, 1, 0) \succ L_2 = (0.10, 0.89, 0.01)$$

$$L_3 = (0, 0.11, 0.89) \prec L_4 = (0.10, 0, 0.90)$$

$$EU(L_1) = u_2 \quad > \quad EU(L_2) = 0.1 \cdot u_1 + 0.89 \cdot u_2 + 0.01 u_3$$

$$EU(L_3) = 0.11 \cdot u_2 + 0.89 u_3 \quad < \quad EU(L_4) = 0.1 \cdot u_1 + 0.9 u_3$$

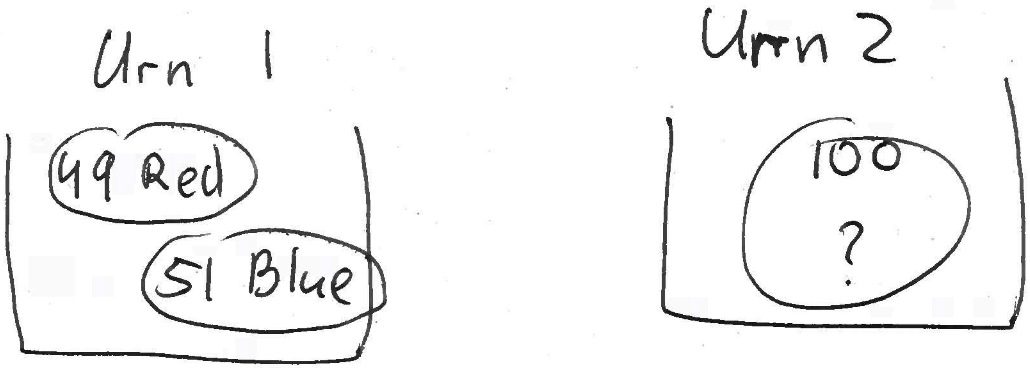
$$+ \qquad + 0.89 u_2 - 0.89 u_3 \qquad\qquad + 0.89 u_2 - 0.89 u_3$$

$$u_2 \quad < \quad 0.1 u_1 + 0.89 u_2 + 0.01 u_3$$

→ reference dependence

→ different attitude towards gains and losses

Ellsberg Paradox



Situation 1

Win 1000 if draw a Red
0 — " — " a Blue

majority chooses urn 1.

Situation 2

Win 1000 if draw a Blue
0 — " — a Red

majority chooses urn 1.

→ Ambiguity aversion

u.f ~ utility function

Monetary outcomes

x monetary outcome

$f : \mathbb{R} \to \mathbb{R}_+$ probability density function (PDF)

$f(x)$ ~ density of x (prob of x if outcomes are discrete)

$(f_1, f_2, \ldots, f_K ; \alpha_1, \alpha_2, \ldots, \alpha_K) \to$ compound lottery

$$f(x) = \sum_{k=1}^{K} \alpha_k f_k(x)$$

f ~ lottery

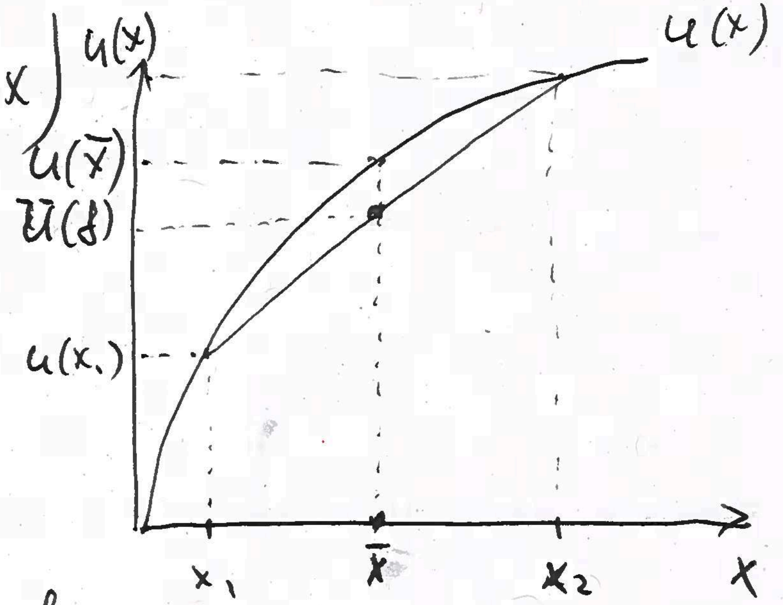
$u(x)$ ~ Bernoulli u.f. (increasing, continuous, bounded)

$$U(f) = \int u(x) f(x) dx$$

vN-M expected utility of lottery f.

Def. A decision maker with a Bernoulli u.f. $u(x)$ is <u>risk averse</u> <u>(strictly)</u> if for every lottery f

$$\int u(x) f(x)\,dx \underset{(<)}{\leq} u\left(\int x f(x)\,dx\right)$$



Ex. $f = (x_1, x_2;\ \tfrac{1}{2}, \tfrac{1}{2})$

$\bar{x} = \tfrac{1}{2} x_1 + \tfrac{1}{2} x_2$

* <u>Risk-loving (strictly)</u> if

$$\int u(x) f(x)\,dx \underset{(>)}{\geq} u\left(\int x f(x)\,dx\right)$$

* <u>Risk-neutral</u> if $\int u(x) f(x)\,dx = u\left(\int x f(x)\,dx\right)$

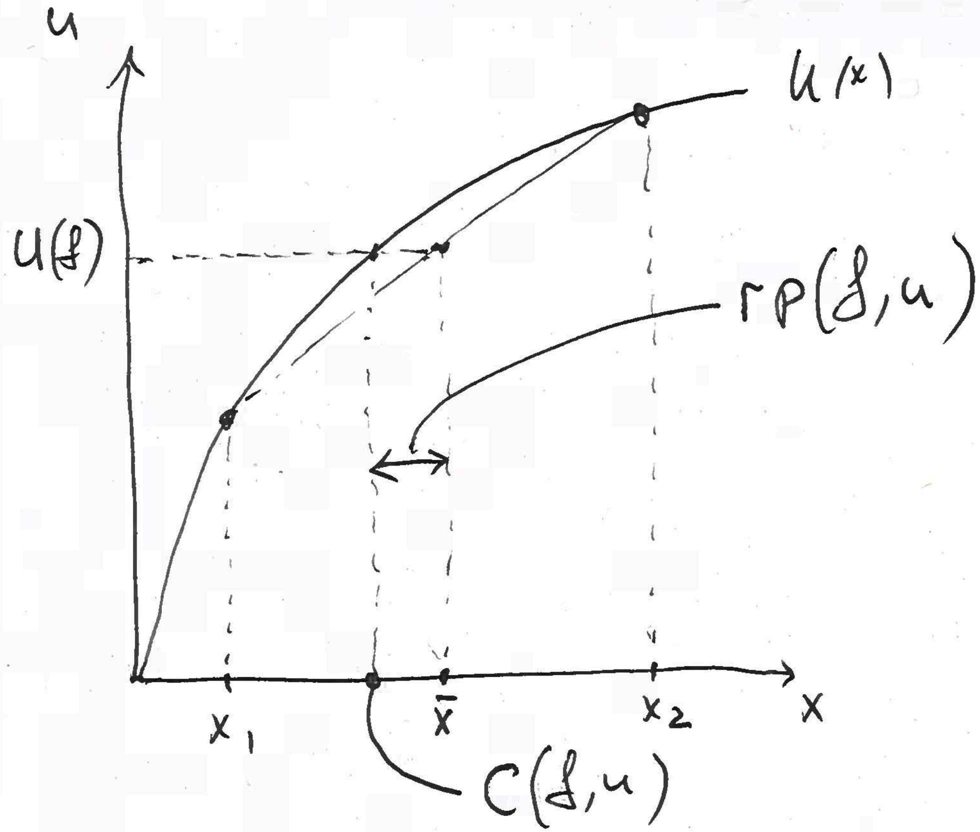
Def. Given $u(\cdot)$, we define the **certainty equivalent** of lottery f, denoted by $C(f,u)$ as the amount of money for which the individual is indifferent between lottery f and the amount of $C(f,u)$:

$$\int u(x) f(x) dx = u(C(f,u)).$$

Def. The **risk premium** of f, denoted by $rp(f,u)$ is

$$rp(f,u) = \underbrace{\int x f(x) dx}_{\text{expected value of } f} - \underbrace{C(f,u)}_{\text{certainty equivalent of } f}$$

($rp(f,u)$ negative if risk-loving)



$$f \sim \left(x_1, x_2 ; \tfrac{1}{2}, \tfrac{1}{2}\right)$$

$$\bar{x} = \int x f(x) dx = \tfrac{1}{2} x_1 + \tfrac{1}{2} x_2$$

Proposition

The following properties are equivalent:

(1) The decision maker is risk averse / risk loving / risk neutral,

(2) $u(\cdot)$ is concave / convex / linear

(3) $c(f,u) \leq / \geq / = \int x f(x) dx$ for all lotteries f

(4) $rp(f,u) \geq 0 / \leq 0 / = 0$ for all f

Insurance

- strictly risk averse DM
- W ~ initial wealth
- D ~ possible loss
- π ~ probability of loss
- q ~ price of insurance, per unit of cover
- α ~ cover (units of insurance bought)

π damage occurs: $$W - D - \underbrace{\alpha \cdot q}_{\text{premium}} + \underbrace{\alpha}_{\text{compensation}}$$

$1-\pi$: $$W - \alpha \cdot q$$

$\alpha = 0$ (no insurance)

π: $$W - D$$

$1-\pi$: $$W$$

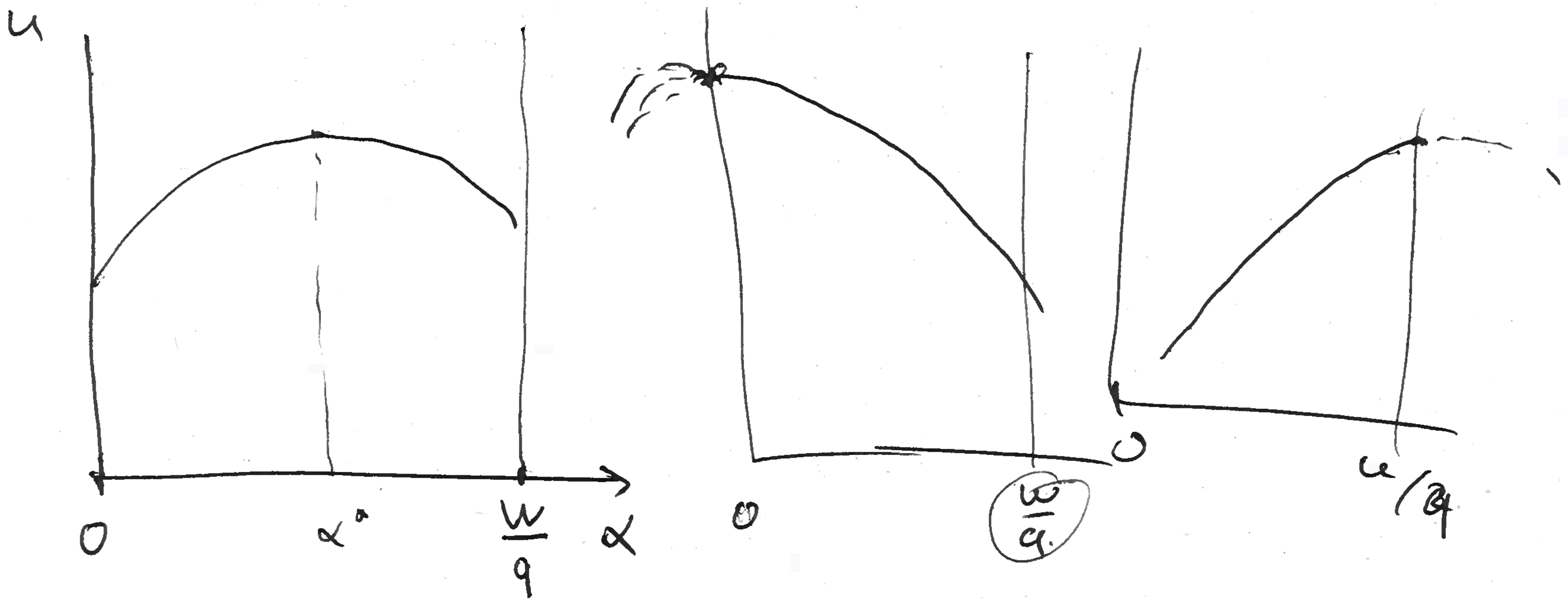
How much cover α to buy?

Assumptions 1. Insurance is actuarily fair

(the insurer makes zero profit, $\underbrace{\alpha q}_{\text{premium}} = \underbrace{\pi \cdot \alpha}_{\text{expected payment}}$)

2. Full cover $\alpha = D$ is affordable,

$D \cdot q < W$.



$\longrightarrow$ From assumption 1, $\boxed{q = \pi}$

K-T conditions for maximization.

$$\max_{\alpha} \quad (1-\pi)\, u\left(W-\alpha\pi\right) + \pi\, u\left(W-\alpha\pi-D+\alpha\right)$$

$$\text{s.t.} \quad \alpha \geq 0, \qquad \alpha\cdot\pi \leq W \iff W-\alpha\pi \geq 0$$

$$\mathcal{L} = (1-\pi)u\left(W-\alpha\pi\right) + \pi\, u\left(W-\alpha\pi-D+\alpha\right) + \mu\cdot\alpha + \lambda\left(W-\alpha\pi\right)$$

F.O.C.

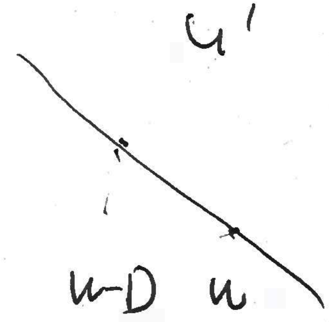
$$\frac{\partial \mathcal{L}}{\partial \alpha} = -\pi(1-\pi)u'\left(W-\alpha\pi\right) + \pi(1-\pi)u'\left(W-\alpha\pi-D+\alpha\right) + \mu - \lambda\pi = 0 \qquad (1)$$

$$\frac{\partial \mathcal{L}}{\partial \mu} = \alpha \geq 0 \qquad (2) \qquad\qquad \frac{\partial \mathcal{L}}{\partial \lambda} = W-\alpha\pi \geq 0 \qquad (3)$$

$$\mu \geq 0 \qquad (4) \qquad\qquad \lambda \geq 0 \qquad (6)$$

$$\alpha\cdot\mu = 0 \qquad (5) \qquad\qquad \lambda\cdot\left(W-\alpha\pi\right) = 0 \qquad (7)$$

<u>Case 1</u> Suppose $\alpha^* = 0 \Rightarrow \lambda^* = 0$ by (7)



By (1) we get

$$\underbrace{\pi(1-\pi)}_{>0}\Big[\underbrace{u'(W) - u'(W-D)}_{<0}\Big] = \underbrace{\mu}_{\geq 0}$$

Cannot be satisfied. $\Rightarrow \alpha^* \neq 0$

<u>Case 2</u> Suppose $\alpha^* = W/\pi \Rightarrow \mu^* = 0$ by (5)

By (1) we get

$$\underbrace{-\pi(1-\pi)}_{<0}\Big[\underbrace{u'(0) - u'\left(\frac{W}{\pi} - D\right)}_{>0}\Big] = \underbrace{\lambda}_{\geq 0}$$

$\frac{W}{\pi} - D > 0$ by Assumption 2.

$\Rightarrow \alpha^* \neq W/\pi$

Case 3 Interior $\alpha \in (0, W/\pi)$

$$\Rightarrow \quad \lambda^* = 0, \quad \mu^* = 0$$

$$(1) \Rightarrow \quad u'(W - \alpha\pi) = u'(W - D - \alpha\pi + \alpha)$$

$$\Longrightarrow \quad W - \alpha\pi = W - D - \alpha\pi + \alpha$$

$$\Longrightarrow \quad \alpha^* = D.$$

$\Rightarrow$ optimal to buy full insurance.

With $\alpha^* = D$:

$$\begin{array}{ll} \pi: & W - D - \alpha^* \cdot \pi + \alpha^* = \underline{W - D\pi} \\ 1-\pi: & W - \alpha^* \cdot \pi = \underline{W - D\pi} \end{array}$$