Microeconomics 1, part B.

Choice under uncertainty

Game Theory

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Choice Under Uncertainty

$C = \{c_1, c_2, c_3, \dots, c_N\}$ outcomes (or consequences)

Def. Given C, a <u>simple lottery</u> L is a list $L = (p_1, p_2, p_3, \dots, p_N)$

with $p_n \geq 0$ for all n and $\sum_{n=1}^{N} p_n = 1$

$p_n \rightarrow$ probability of outcome n

$\mathcal{L}(C) \rightarrow$ the set of all possible simple lotteries over C

$$\mathcal{L}(C) = \underset{\uparrow \text{simplex}}{\Delta}(C) = \left\{ p \in \mathbb{R}^N_+ : \sum_{n=1}^{N} p_n = 1 \right\}$$

<u>$N-1$ dimensional</u>

0-dim simplex $\quad p_1 = 1$

point

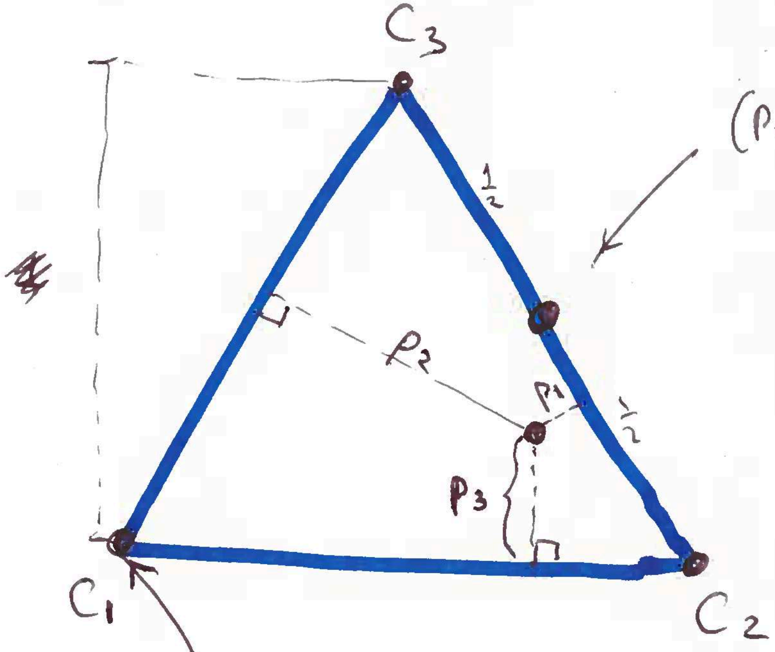
1-dim simplex $\quad p_1 + p_2 = 1$

$\left(p_1 = \frac{2}{3}, \; p_2 = \frac{1}{3}\right)$

2-dim simplex

$p_1 + p_2 + p_3 = 1$

$(p_1, p_2, p_3) = \left(0, \frac{1}{2}, \frac{1}{2}\right)$



Degenerate lottery = lottery that has $p_n = 1$

Compound lotteries

Def. Given K simple lotteries over C
$L_k = (p_1^k, p_2^k, \dots, p_N^k)$ and probabilities
$(\alpha_1, \alpha_2, \dots, \alpha_K)$ with $\alpha_k \geq 0$ and $\sum \alpha_k = 1$,
the compound lottery $(L_1, \dots, L_K; \alpha_1, \dots, \alpha_K)$
is a lottery that yields each simple lottery L_k with probability α_k.

Reduced lottery L of a compound lottery
$(L_1, \dots, L_K; \alpha_1, \dots, \alpha_K)$ is a lottery $(p_1, \dots, p_N)$
that yields the same distribution over outcomes:

$$p_n = \alpha_1 \cdot p_n^1 + \alpha_2 \cdot p_n^2 + \dots + \alpha_K p_n^k.$$

so $L = \alpha_1 L_1 + \alpha_2 L_2 + \dots + \alpha_K L_K$

Example

$N=3, \quad C=(c_1, c_2, c_3)$

$L_1 = (1, 0, 0)$

$L_2 = \left(\frac{1}{4}, \frac{3}{8}, \frac{3}{8}\right)$

$L_3 = \left(\frac{1}{4}, \frac{1}{8}, \frac{5}{8}\right)$

Compound lottery

$\left(L_1, L_2, L_3; \frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right)$

$p_1 = 1 \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{1}{6} = \frac{5}{8}$

$p_2 = \ldots$

$p_3 = \ldots$

$L = (p_1, p_2, p_3) = \left(\frac{5}{8}, \frac{7}{48}, \frac{11}{48}\right).$

Preferences $\succsim$ ($\succ$, $\sim$)

Axiom 1 Completeness For all $L, L' \in \mathcal{L}$,

either $L \succsim L'$ or $L' \succsim L$ (or both, $L \sim L'$)

Axiom 2 Transitivity For all $L, L', L'' \in \mathcal{L}$

if $L \succsim L'$, and $L' \succsim L''$, then $L \succsim L''$.

Axiom 3 Continuity For all $L, L', L'' \in \mathcal{L}$

such that $L \succsim L'' \succsim L'$, then there exists $\alpha \in [0, 1]$

such that $\underbrace{\alpha \cdot L + (1-\alpha) \cdot L'}_{\text{compound lottery.}} \sim L''$.

Axioms 1-3 $\Rightarrow$ There exists a utility function u that represents $\succsim$

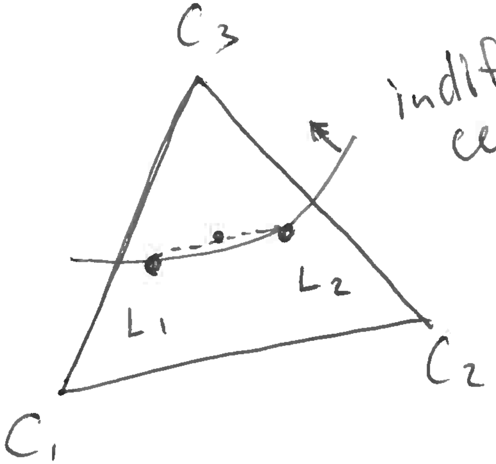
$$L \succsim L' \iff u(L) \geq u(L').$$

<u>Axiom 4</u> <u>Independence</u>

For all $L, L', L'' \in \mathcal{L}$ and all $\alpha \in (0,1)$

$$L \succsim L' \iff \alpha L + (1-\alpha) L'' \succsim \alpha L' + (1-\alpha) L''.$$

<u>Claim</u> If $\succsim$ satisfy Independence, then the indifference curves are parallel straight lines.

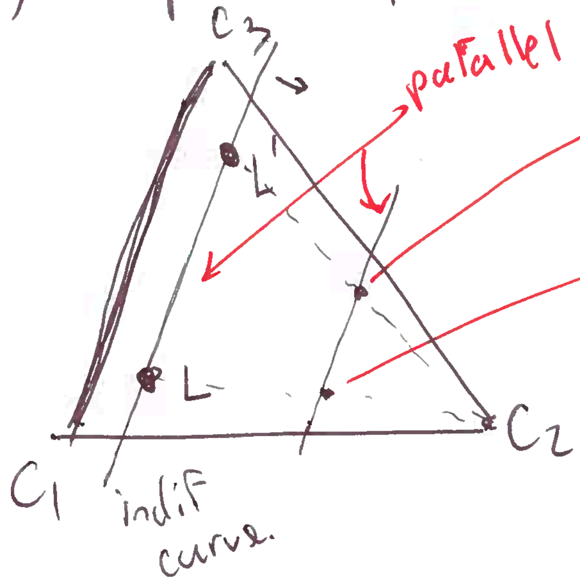


$\frac{1}{2} L_1 + \frac{1}{2} L_2 \sim$ by Independence

$\frac{1}{2} L_2 + \frac{1}{2} L_2 = L_2$

$\Rightarrow$ indif. curves are straight lines

2) parallel straight.



$$\frac{1}{2}L+\frac{1}{2}C_2 \sim \frac{1}{2}L'+\frac{1}{2}C_2$$

by Independence.

<u>Def</u> Given $C=\{c_1,\dots,c_N\}$, the utility function $U:\mathcal{L}(C)\to\mathbb{R}$ has the expected utility form if there is an assignment of numbers $u=(u_1,\dots,u_N)$ such that for every lottery $L=(p_1,\dots,p_N)$ we have

$$\underline{U(L)}=p_1u_1+p_2u_2+\dots+p_Nu_N$$

von Neumann-Morgenstern (vN-M) utility function

$u: C\to\mathbb{R}$ Bernoulli utility function.

__Theorem__ (Expected Utility Theorem)

Suppose that a preference relation $\succsim$ satisfies Axioms 1–4. Then, and only then, $\succsim$ admits a vN-M expected utility representation.

Given $L = (p_1, \dots, p_N)$ and $L' = (p_1', \dots, p_N')$

we must have

$$L \succsim L' \iff \sum_{n=1}^{N} p_n u_n \geq \sum_{n=1}^{N} p_n' u_n,$$

where $(u_1, \dots, u_N)$ is the Bernoulli utility of outcomes $(c_1, \dots, c_N)$

Proposition Suppose $U: \mathcal{L} \to R$ is a vN-M utility function for a preference relation $\succsim$.
Then, and only then, for each $\beta > 0$ and each $\gamma \in R$,

$\tilde{U}(L) = \beta U(L) + \gamma$ represents the same $\succsim$.

I.e. for all $L, L' \in \mathcal{L}$

$$U(L) \geq U(L') \iff \tilde{U}(L) \geq \tilde{U}(L').$$

$c_1 \succ c_2 \succ c_3$ ordinal

~~...~~ $u_1 > u_2$ cardinal.

$u_1 - u_2$ compare to $u_2 - u_3$