Section B

Question 1. Brendan is an expected utility maximizer with the following Bernoulli utility function:

$$u(x) = \sqrt{x}.$$

Brendan holds a portfolio of stocks of local companies. The local economy can go either up or down. If the economy goes up, then the portfolio will be worth 2000. If the economy goes down, these stocks will be worthless, so the portfolio will be worth zero. The probability that the local economy will go bad is 0.2. Brendan also has some safe assets which are worth W no matter what happens.

(a) Calculate the certainty equivalent of Brendan's current holdings as a function of W. How would this certainty equivalent change if Brendan's Bernoulli utility function were linear in his wealth (instead of $\sqrt{x}$)?

$$EU = 0.2\sqrt{W} + 0.8\sqrt{2000 + W}$$

$$\sqrt{CE} = 0.2\sqrt{W} + 0.8\sqrt{2000 + W}$$

$$\Leftrightarrow CE = \left(0.2\sqrt{W} + 0.8\sqrt{2000 + W}\right)^2$$

$$\Leftrightarrow CE = 0.04W + 2 \times 0.2 \times 0.8\sqrt{W(2000 + W)} + 0.64(2000 + W)$$

$$\Leftrightarrow CE = 1280 + 0.68W + 0.32\sqrt{W(2000 + W)}$$

If Bernoulli utility function was linear, the CE would be equal to the expected value of the holdings:

$$CE = 0.2 \times W + 0.8 \times (2000 + W) = W + 1600$$

Suppose now that Brendan can buy insurance. One unit of insurance costs p pounds, and pays 1 pound in the event that the stocks become worthless. Brendan can buy any nonnegative number of units of insurance, but he cannot spend more than his wealth W on insurance. Use letter q to denote the number of insurance units bought by Brendan, and assume $W > 1000$.

(b) Set up Brendan's expected utility maximization problem and write out the Kuhn-Tucker conditions that characterize the optimal number q^* of insurance units he should buy.

$$\max_{q} \quad 0.2 \times \sqrt{W + q - qp} + 0.8 \times \sqrt{W + 2000 - qp}$$

subject to

$$\begin{aligned} q &\geq 0 \quad (A) \\ qp &\leq W \quad (B) \end{aligned}$$

The Lagrangian can then be written as

$$\mathbb{L} = 0.2 \times \sqrt{W + q - qp} + 0.8 \times \sqrt{W + 2000 - qp} + \mu q - \lambda(qp - W)$$

where $\lambda \geq 0$ and $\mu \geq 0$ are the Lagrange multipliers. The first order conditions (FOCs) are given by:

$$\frac{\partial \mathbb{L}}{\partial q} = 0.2(1-p)\frac{1}{2\sqrt{W + q - qp}} - 0.8p\frac{1}{2\sqrt{W + 2000 - qp}} + \mu - \lambda p = 0 \quad (1)$$

$$\frac{\partial \mathbb{L}}{\partial \mu} = q \geq 0 \;\; (2); \quad \mu \geq 0 \;\; (3); \quad q\mu = 0 \;\; (4)$$

$$\frac{\partial \mathbb{L}}{\partial \lambda} = W - qp \geq 0 \;\; (5); \quad \lambda \geq 0 \;\; (6); \quad \lambda(W - qp) = 0 \;\; (7)$$

Consider for the rest of the problem the case of actuarially fair insurance.

(c) How is the price p of a unit of insurance defined for the case of actuarially fair insurance? What are the profits of the insurance company in this case?

Under actuarially fair insurance, the insurance company makes zero profits. Let α be the probability that the economy is bad, $\alpha = 0.2$. The profit is

$$(p - \alpha \cdot 1)q = 0.$$

Thus, the actuarily fair per-unit premium is $p = \alpha = 0.2$.

(d) How many units of insurance will Brendan buy (that is, what is the value of q^*)? *Hint: Use the Kuhn-Tucker conditions to show that* $q^* \in (0, \frac{W}{p})$, *i.e. the solution is interior, and then to calculate* q^*.

Consider first an interior solution, $q^* \in (0, W/p)$. The constraints are not binding, and the Lagrange multipliers are zero, so $\mu^* = \lambda^* = 0$. Thus, (1) becomes:

$$\frac{\partial \mathbb{L}}{\partial q} = 0.2 \times 0.8\frac{1}{2\sqrt{W + 0.8q}} - 0.8 \times 0.2\frac{1}{2\sqrt{W + 2000 - 0.2q}} = 0$$

which implies

$$\sqrt{W + 0.8q} = \sqrt{W + 2000 - 0.2q} \Leftrightarrow q^* = 2000.$$

Indeed, if insurance is actuarially fair, the risk-averse decision maker is fully insured.

Let us now use the Kuhn-Tucker conditions to show that $q^* = 0$ and $q^* = W/p$ are not solutions.

Suppose $q^* = 0$. Then, by (7) it must be that $\lambda^* = 0$. Substituting $\lambda^* = 0$, $q^* = 0$, and $p = 0.2$ to the FOC w.r.t. q, we obtain

$$\frac{\partial \mathbb{L}}{\partial q} = 0.2 \times 0.8 \frac{1}{2\sqrt{W}} - 0.8 \times 0.2 \frac{1}{2\sqrt{W + 2000}} + \mu = 0$$

Thus,

$$\mu = 0.08\Big(\frac{1}{\sqrt{W + 2000}} - \frac{1}{\sqrt{W}}\Big).$$

The expression in parenthesis is negative, and hence $\mu < 0$, which is a contradiction to (3). Therefore, $q^* = 0$ is not a solution.

Suppose $q^* = \frac{W}{p}$. Then, by (4), it must be that $\mu^* = 0$. Substituting $\mu^* = 0$, $q^* = \frac{W}{p}$, and $p = 0.2$ in (1) we obtain:

$$\frac{\partial \mathbb{L}}{\partial q} = 0.2 \times 0.8 \frac{1}{2\sqrt{\frac{W}{0.2}}} - 0.8 \times 0.2 \frac{1}{2\sqrt{2000}} - 0.2\lambda = 0$$

Thus,

$$\lambda = 0.8 \left(\frac{1}{2\sqrt{\frac{W}{0.2}}} - \frac{1}{2\sqrt{2000}} \right)$$

Because $W > 1000$, we have $W/p = W/0.2 > 5000$, so the expression is parenthesis is negative, which in turn implies $\lambda < 0$. This is a contradiction to (6). Hence, we conclude that $q^* = \frac{W}{p}$ is not a solution.

Question 2. Consider the following extensive form game between two co-workers named Ken and Barbie.

Stage 1. Barbie (Player 1) decides whether to start a project or not. If not, then the game ends with both players getting 0. Otherwise, move to stage 2.

Stage 2. Ken (Player 2) decides to contribute to the project or to free ride. If Ken contributes, then the game ends with both players getting 1. Otherwise, move to stage 3.

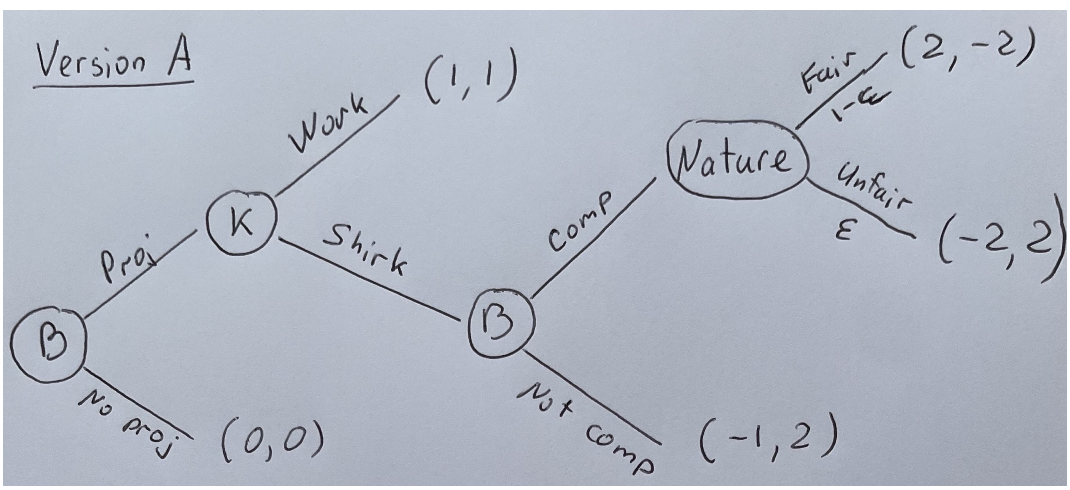
Stage 3. Barbie decides whether to file a complaint to the line manager about Ken's lack of effort. If she does not complain, then the game ends with the payoffs -1 for Barbie and 2 for Ken. If she does complain, then there is a small probability $\varepsilon = 0.1$ that the line manager is prejudged against Barbie. If the line manager is fair, then Ken will be punished for his lack of effort, and the payoffs will be 2 for Barbie and -2 for Ken. However, if the line manager is prejudged, then Ken will be rewarded and Barbie punished, and the payoffs will be -2 for Barbie and 2 for Ken.

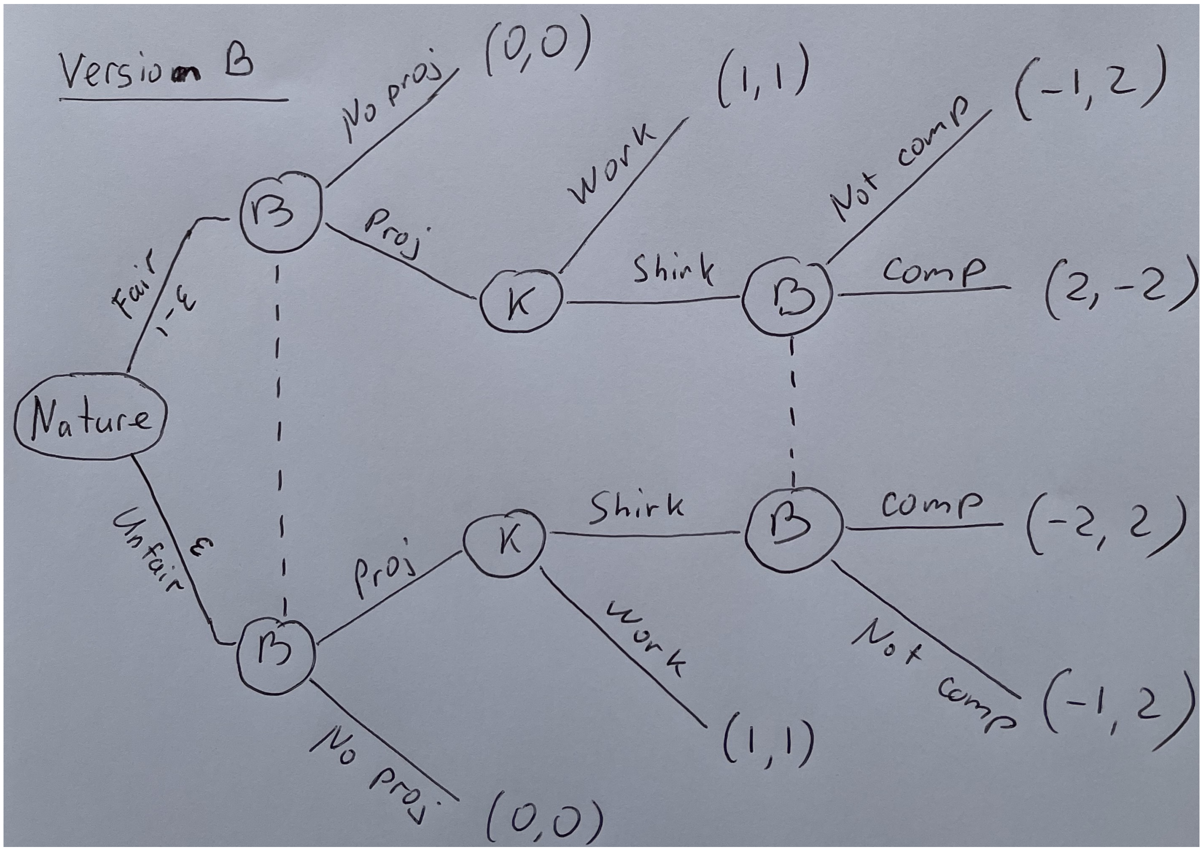
Consider two versions of this game.

In **version A**, both players have the same uncertainty about whether or not the line manager is prejudged.

In **version B**, only Barbie is uncertain about the line manager's attitude, whereas Ken knows exactly whether or not the line manager is prejudged.

(a) Depict versions A and B of this extensive-form game graphically, using tree diagrams. In what way are these diagrams different?





In version A, the players are symmetrically informed. Therefore, the uncertainty about the manager's attitude manifests as a lottery over the outcome where Barbie chooses to complain.

In version B, the players are asymmetrically informed. Ken is iformed about the manager's attitude but Barbie is not. This is a proper game of incomplete information, where Barbie's lack of information which decision node she is at is shown as dashed lines (information sets).

(b) Can backwards induction can be used to solve version **A**? Explain briefly. Apply an appropriate equilibrium concept and find an equilibrium in pure strategies for version **A**. Please clearly describe the strategies (and beliefs, if needed) of the players, and argue why this is an equilibrium.

Version A is solvable by backward induction. The lottery of $(2,-2)$ and $(-2,2)$ with probabilities $(1-\varepsilon,\varepsilon)$ are replaced with their expected payoffs. So, when Barbie complains, the expected payoffs are

$$\Big(2(1-\varepsilon)+(-2)\varepsilon,(-2)(1-\varepsilon)+2\varepsilon) = (2-4\varepsilon,-2+4\varepsilon).$$

Then let us derive a subgame perfect equilibrium using Backward Induction. Barbie complains (because $2-4\varepsilon = 2-0.4 > -1$, Ken works (because

$1 > -2 + 4\varepsilon = -2 + 0.4$), and Barbie starts the project (because $1 > 0$). The subgame perfect equilibrium is

Barbie: (Proj, Comp); **Ken:** Work, with the payoffs $(1, 1)$.

(c) Can backwards induction can be used to solve version **B**? Explain briefly. Apply an appropriate equilibrium concept and find an equilibrium in pure strategies for version **B**. Please clearly describe the strategies (and beliefs, if needed) of the players, and argue why this is an equilibrium.

Version B is not solvable by backward induction, because it does not have any subgames other that the game itself.

A perfect Bayesian equilibrium in pure strategies is as follows. Barbie does not complain. Anticipating this, Ken prefers to shirk, regardless of his information about the manager. Anticipating Ken's shirking, Barbie prefers not to start the project in the first place. To complete the description of PBE, we specify Barbie's beliefs. In her first information set, her beliefs are given by the prior, $(1-\varepsilon, \varepsilon)$. Her second information set is not reached, so Bayes' rule does not determine the beliefs. Let the beliefs $(1 - \beta, \beta)$ in that information set be such that β is large enough, so that Barbie's best response would be not to complain:

$$2(1 - \beta) + (-2)\beta < -1. \qquad (1)$$

Barbie: (No Proj, $(1 - \varepsilon, \varepsilon)$; Not Comp, $(1 - \beta, \beta)$), where β satisfies (1).
Ken: Shirk, regardless of the information about the manager.
The equilibrium payoffs are $(0, 0)$.

Methodological Note: To find a PBE in pure strategies, observe that there are only two choices that Barbie can make in her second information set. One is to complain, and the other is not to complain. Suppose first that Barbie complains. Then, anticipating Barbie's choice, Ken works if the manager is fair but shirks if the manager is unfair. Anticipating that Ken works with probability $1-\varepsilon = 0.9$, Barbie chooses to start the project. But then, by Bayes' rule, Barbie's beliefs in the second information set must be that if Ken shirks, it must be the case that the manager is unfair. Thus, Barbie's best response should be not to complain. We conclude that there is no PBE where Barbie complains. This leaves us with the only other case that Barbie does not complain, and we proceed as above.

(d) Draw economic conclusions about how uncertainty and asymmetric information of coworkers may influence incentives for teamwork.

- Asymmetric information presents additional challenges for providing incentives for teamwork, as a player with superior information has different incentives depending on his information.

- Giving superior information to one player need not be desirable as it may destroys incentives for teamwork. Information influences decisions in a complex way. As incentives are different depending on information, behavior is also different, which in turn affects the choices of the other, uninformed player.