Search and Matching - Ph.D. Training Course
Lecture 1: Frictionless Non-Transferable Utility Matching

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December 6th 2013

Course Overview

The course has four parts:

1. Matching: Frictionless and no transfers
2. Matching: Frictionless with transfers
3. Search: Basics of Search Models
4. Search and Matching combined

Goals:

- To understand "what is out there"
- To know where to start to learn more
- To get a feeling for what is missing

Lecture notes:
http://homepages.econ.ed.ac.uk/~pkircher/searchmatching-PhD-lecture-i.pdf (with i=1,2,3,4)

Who we are and our interests

Introduce ourselves and our interests:

a) Micro Theory
b) Applied Labor
c) Macro
d) Other

Lecture 1: Two-Sided Frictionless One-to-One Matching, No Transfers

Examples:

- Men and Women
- School (slots) and Pupils
- Hopital (positions) and Residents?
- Econ Departments and Job Market Candidates?

Questions:

- Does one-to-one make sense (many-to-one is like one-to-one if there are no "externalities")
- Does the assumption that transfers are absent or fixed make sense?

The Setup

The "Players":

- Set of Men (or firms): $M = \{m_1, m_2, ..., m_{|M|}\}$
- Set of Women (or workers): $W = \{w_1, w_2, ..., w_{|W|}\}$
- They are different people: never $m_i = w_j$

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Preferences: Each player strictly ranks people on the other side of the market.

- Say that $w_1 \succ_{m_1} w_2$ if men m_1 strictly prefers w_1 to w_2.
- Say that $m_1 \succ_{m_1} w_2$ if men m_1 strictly prefers being single to w_2.
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Players and Preferences together form a marriage market.

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Preferences can be represented by a utility function: $u_{m_1}(w_1) > u_{m_1}(w_2)$ iff $w_1 \succ_{m_1} w_2$. Importantly: difference in utility is meaningless.

Definition of a Matching

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Let v be the function that assigns to each women the corresponding men or herself (being single):

- $v(w_i) = m_j$ iff $\mu(m_j) = w_i$
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Explain matching based on "types" in words.

Stability

A matching μ and associated v is called stable if

- Individual rationality: every matched person prefers their parter over being single ($\mu(m_i) \succeq_{m_i} m_i$ for all i and $v(w_j) \succeq_{w_j} w_j$ for all j) and

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- Explain in words the difference to transferable utility matching.

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- Move to t+1 with the same rules as in t, unless no offer has been made in t, in which case terminate the algorithm and all matches become final.

Existence of Stable Matching: Gale-Shapley Algorithm

Insights:

- Terminates in finite time (at most $|M||N|$ rounds)
- Is stable:
  - individual rational? yes, no partners lower than being single
  - pairwise stability? Assume not. Then there exists (m, w) s.t.: $\mu(m) \neq w$ but $w \succ_m \mu(m)$ and $m \succ_m v(m)$ But then m must have proposed to w in earlier rounds, and since w is not matched to him she must be matched with someone better, contradicting the second part.

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  - Example.
  - Stability does not hold for single-sex roommate problem (example).

Additional Insights

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- Men-proposing algorithm is preferred by all men over any other stable matching
  (Women-proposing algorithm is preferred by all women over any other stable matching.)
- It is dominant strategy for men to reveal their true preferences.
- It is not dominant strategy for the women to reveal their true preferences (and there is no mechanism that allows truthful revelation by both sides)
- Current: Incentives to lie for women disappear in large markets (current research, see Azevedo, Budish...)

Applications

Applications:

- Medical Residency Matching (Al Roth book)
  - current: couples
  - not everyone knows each other (how does the meeting work)?
  - transfers and collusion (Hatfield and others)
- Similar ideas in other areas (Kidney exchange, Roth...)

Applications

Applications: School Choice:

- schools not as "players" but as "priorities"
- alternative to the "Boston Mechanism" (everyone hands in preference list, everyone gets their first choice if possible (no difference), of the students and sets left everyone gets their second choice, etc).

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- advantage of GS: strategy proof, best stable match
- disadvantage of GS: not Pareto efficient (see serial dictatorship)
- advantage of Boston: reflects intensity of preferences (recent AER)
- there are more general approaches (mech design, top trading cycles, tie-breaking)
- To do: peer effects, general market design, modeling why money is not used