Marginal Subsidies in Tullock Contests*

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Abstract

In a general Tullock contest, we examine a situation where a limited resource can be used to provide marginal subsidies to either player (weak or strong), or to increase the prize directly. We show that to maximize total effort, subsidizing the weak/strong player is preferred when the contest is sufficiently accurate/inaccurate. This result generalizes to n-player lottery contests. In a lottery contest (Tullock contest with $r = 1$), we derive the optimal scheme for a full range of resource: when the resource is small, it is optimal to only subsidize the weak player; when it is large, both players should be subsidized simultaneously.

Key words: Tullock contest, resource, marginal subsidy, prize.

JEL Classifications: C7, D7, O3.

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1 Introduction

A contest is a situation in which players compete against each other by making irreversible effort, often for a prize or multiple prizes. Many situations in the real world have been studied as contests or "contest situations".[1] In practice, setting prizes has been considered as the most important and effective way to attract potential contestants and stimulate competition between participants. As a result, prize allocations have been studied extensively in the contest literature.[2]

In addition to awarding prizes, subsidizing contestants can also be a good way to induce effort. This has not drawn much attention in the literature, although in practice subsidies are often observed in contests or contest situations. By estimating an econometric model using contractor-level data, Lichtenberg (1990) shows that the US Department of Defense (DoD) encourages private military R&D investment not only by establishing prizes, but also by subsidizing expenditures (costs of making effort) dedicated towards winning the prize.[3,4] He concludes: "On the surface, it appears that the *marginal subsidy* on the R&D investment is zero, but this is only true in the short term. Due to the DoD's policy of allowable-cost determination, the long-run marginal subsidies are substantial."

Similarly, within large firms in sectors where product innovation is of importance, there may be two or more teams (or individuals) working independently on the same project or task (e.g., designing next-generation products). The best-performing team will be rewarded with a prize, such as a bonus or opportunity for promotion. At the same time, the firm would typically provide resources to reduce the costs of the teams in carrying out their tasks. The question we address is whether it is better for the firm to provide such a marginal subsidy, or better to make the prize larger. Other possible applications include education. For instance,

[1] Contest situations refer to a variety of interactions in reality such as sports, rent-seeking, litigation, beauty contests, patent races, research and development (R&D), political competition, arms races, etc.

[2] For instance, Clark and Riis (1998), Moduvanu and Sela (2001), Szymanski and Valletti (2005), Fu and Lu (2009, 2012), Akerlof and Holden (2012), Schweinzer and Segev (2012) and among others have been studied prize allocation (or some related issues) within different settings.

[3] Lichtenberg (1988) shows that the DoD has conventionally sponsored numerous design competitions to stimulate private investment in defense technology. R&D contests sponsored by the DoD remain common. For instance, in 2007, the DoD set a prize of 1 million dollars to lessen the weight of more than 20 pounds of batteries a solider carries on a typical four-day mission.

[4] This and other examples have been discussed in Fu, Lu and Lu (2012).

in a class where students exert effort to achieve higher degree classifications, it is common for a teacher to offer *marginal help* to a student or a specific group of students: the teacher will offer more help when a student exerts more effort.[5]

The contest designer may face a budget constraint on the resource that can be used as subsidies. For instance, the DoD or the firm may have a fixed amount of money available for providing subsidies, the teacher may have a fixed amount of time for tutoring her students. Following convention in the contest theory literature where a (fixed) prize is often assumed to have no intrinsic value to the contest designer, we assume that the (limited) resource also has no intrinsic value to the contest designer.[6] Then the problem for the contest designer is: how can the limited resource be used most efficiently to maximize total effort. For instance, the DoD, which has the objective of encouraging military R&D in some specific field, has to decide which contractor to subsidize, the "underdog" (the weak firm) or the "favorite" (the strong firm); similarly, the firm has to decide which team to subsidize, the strong team or the weak team; the teacher, who wants to improve the overall academic performance of her students, may have to decide whether the helpdesk is mainly for helping the less able or more able students. Moreover, if feasible, would adding the resource directly to the prize be more efficient than providing subsidies? For example, should the DoD (firm) use the money to subsidize a contractor (team) or add the money directly to the prize? This paper is an attempt to answer the above questions.

Our analysis is in the context of Tullock contests.[7] Notice that the parameter r in a Tullock Contest Success Function (CSF), which is often referred to as the discriminatory power or accuracy level of the contest.[8] In this paper, as is common in the literature, Tullock

[5]For instance, suppose a teacher runs a helpdesk in order to help some students. A student who does little homework (i.e., makes little effort) will gain little from the helpdesk (i.e., gets little help); while a student who is well prepared (i.e., makes a large effort) will benefit a lot from it (i.e., gets much help).

[6]While this might often seem unrealistic, focusing on the question of utilizing a fixed resource most efficiently can be regarded as the first stage in a two-stage process: at the second stage, not analyzed here, using the cost function (of raising the resource) and the revenue function (of effort) derived from the results we establish, the optimal amount of the resource can be determined.

[7]Skaperdas (1996) shows that the Tullock CSF is the only continuous success functional form which satisfies several easily interpretable axioms.

[8]Clark and Riis (1996) show that the Tullock CSF can be interpreted as the outcome of a model in which each player's effort is evaluated with an error where the variance of the error is reflected in the parameter r. This justifies the interpretation of r as accuracy of the contest. In particular, Wang (2010) specifically interprets r as the contest's accuracy level.

contests with $r = 1$ are referred to as "lottery contests".

In a general Tullock contest we look at a situation where the contest designer has a sufficiently small amount of resource s to be used to provide marginal subsidies to either player or to increase the prize directly. We can break the effect of a subsidy into two components: a direct effect and an indirect effect. For the direct effect—the response of the player being subsidized—is that though either player would increase effort if subsidized, the effect of the subsidy on the recipient's effort is greater (smaller) if applied to the stronger player when $r < 1$ ($r > 1$).[9] The indirect effect—how the other player responds—is that the weak player decreases effort when his opponent is subsidized, while the strong player increases effort when his opponent is subsidized. We show that the overall effect depends on the accuracy level of the contest: in order to maximize total effort, subsidizing the strong player is preferred when the contest is sufficiently inaccurate (when $r < r^*$), while subsidizing the weak player is preferred when the contest is sufficiently accurate (when $r > r^*$). When the ability difference becomes larger, subsidizing the weak player is more likely to be preferred (r^* decreases with the ability difference). Moreover, increasing the prize is always dominated by the preferred-subsidy-scheme.

Intuitively, from the angle of "individual efficiency", the strong player should be subsidized as he is more efficient in exerting effort, which is the main reason why the direct effect favors subsidizing the strong player when $r < 1$. However, from the angle of "competitive balance", the weak player should be subsidized as it will make the contest more competitively balanced and thus stimulates competition, which explain why the strong (weak) player makes more (less) effort when the weak (strong) player is subsidized. Roughly speaking, when the contest is sufficiently noisy ($r < r^*$), the competition is not fierce, "individual efficiency" outweighs "competitive balance" and the strong player should be subsidized. The reverse occurs when the contest is sufficiently accurate ($r > r^*$).

In an n-player lottery contest (i.e., a Tullock contest with $r = 1$), we show that it is strictly better to subsidize the weakest player, which is in line with our previous finding in a Tullock contest. In addition, in a two-player lottery contest, we allow $s \in (0, +\infty)$ and derive the optimal scheme that maximizes total effort: when the resource is relatively small

[9] We model a player's "strength" by his marginal cost of exerting effort, with the "strong" player referring to the one with the smaller marginal cost.

($s \leq \bar{s}$), it is optimal to only subsidize the weak player, as our earlier results imply; when the resource is relatively large ($s > \bar{s}$), it is optimal to subsidize both players simultaneously.

In our model, providing a marginal subsidy to a player requires that the contest designer is able to verify the recipient's effort level. In cases where effort can be fully verified, theoretically, it is easy for the contest designer to construct a simple contract that implements the maximal individually rational effort and extracts all surplus from the player. Thus, a contest may be suboptimal.[10] One reason why contests with subsidies are used (rather than contracts) could be that effort itself is not perfectly observable but instead a noisy signal, such as final quality of output, is.[11] However it may be that some complementary inputs are observable, and these can be subsidized. For example, the DoD's audits should be able to verify how much physical investment a firm has made on the research project, which can be subsidized.[12] Finally, it may be that effort is observable but not contractible, so that contests can be a good way to induce effort. For example, assume the teacher can observe a student's effort with little cost, but a contract between the teacher and a student is not feasible as effort is not verifiable. However, the teacher may be able to commit to offering help to a student, and the student receives more help the more effort that is exerted—for example the more pieces of work she hands in.

In this paper we focus on linear subsidies. It is straightforward however to show that non-linear schemes can be more effective in increasing total effort, such as a scheme where the subsidy is delivered only if a player's effort is above some threshold. There are a number of reasons why we focus on linear subsidies. First, the linear case is a natural benchmark case that yields sharp results.[13] Second, contests with linear subsidies are observed in practice (see the DoD example discussed above). Third, a linear scheme is likely to be much simpler to implement.

When the subsidy is linear, the cost of exerting each unit of effort can be subsidized

[10] Notice that a similar issue arises elsewhere in the literature on contests with subsidies (reimbursements), e.g., Cohen and Sela (2005), Matros and Armanios (2009), Matros (2012), etc. These papers will be discussed in the next section on related literature.

[11] Clark and Riis (1996) show that the Tullock CSF can be interpreted as the outcome of a noisy performance ranking model in which each player's output consists of an effort term and a noise term. Fu and Lu (2012) generalize Clark and Riis's (1996) model to a multi-prize case.

[12] Although the DoD cannot verify how much unobservable effort (e.g., psychic effort) the firm has done, the incentive to win the contest ensures complementary unobservable effort will be made by the firm.

[13] As, for example, linear (i.e., piece rate) payment contracts are regularly studied in contract theory.

according to a pre-specified constant proportion as this effort is being made, regardless of the previous effort level. However, when a non-linear subsidy scheme is used, the subsidy for the next unit of effort depends on the current (cumulative) effort level, which may be difficult to determine, especially when a contestant's effort consists of multiple components that are being made simultaneously.[14] An easy way of solving this complication of course is to provide subsidies when the tournament is over, i.e., when players' final effort levels are realized. However this may raise a commitment issue as the contest designer will have an incentive to renege on any subsidy as effort is sunk. In addition, even without the above commitment issue, a non-linear subsidy scheme which provide subsides ex post may yield less effort when the recipient's (cost) budget is constrained during the contest.

2 Related Literature

Despite the voluminous literature that has grown from Tullock's (1980) seminal work,[15] there are only a few papers studying contests with subsidies (or reimbursement) in the literature on imperfectly discriminating contests. Two papers consider lottery contests (i.e., Tullock contests with $r = 1$) with full reimbursement: Cohen and Sela (2005) show that if the winner's cost of effort is fully reimbursed, there is a unique internal equilibrium where the weak player wins with the higher probability.[16] Matros (2012) analyzes the n-player model and discusses the properties of the pure-strategy equilibria. In addition, Matros and Armanios (2009) consider reimbursements in a general[17] Tullock contest with homogeneous players and find that the winner-reimbursed-contest maximizes net total spending while the loser-reimbursed-contest minimizes it.[18] Notice that in the above papers, contestants' equilibrium effort levels must be assumed to be common knowledge, otherwise, the contest designer

[14] For instance, in a research contest such as that sponsored by the DOD, a competing firm may need to allocate a variety of different inputs to a project, and determining the order in which these "efforts" are made in order to fully determine what the current (cumulative) effort level is may be costly or impossible.

[15] For general surveys of contests, see Nitzan (1994), Szymanski (2003), Corchon (2007), Congleton et al. (2008) and Konrad (2009).

[16] The "internal equilibrium" is such that players' strategies are derived from the interior solution that solves the first-order conditions. Besides the internal solution, there are corner solutions in which one of the contestants chooses to stay out of the contest.

[17] That is, where r can take any value provided a pure-strategy equilibrium exists.

[18] Kaplan et al. (2002) provide several examples of contests with reimbursements in both politics and economics.

cannot provide reimbursements accurately. In this paper, we maintain this assumption.[19]

Our settings are different from the above research works mainly in three ways. First, we consider a Tullock contest with heterogeneous players. Second, in our model a player is subsidized regardless of who wins eventually, i.e., subsidy is not contingent. Third, we assume there is a limited amount of resource which can be used for subsidies and we compare the efficiencies of subsidizing different players.

Contests with reimbursements have also been studied in the literature on auctions or perfectly discriminating contests. Riley and Samuelson (1981) introduce the "Sad Loser Auction" where the winner gets his bid back. Goeree and Offerman (2004) analyze the Amsterdam auction in which the highest losing bidder obtains a premium which depends on his own bid. Clark and Riis (2000) show that an official who obtains bribes (effort) from firms (players) will favor the firm who is more likely to value the prize less. Che and Gale (2003) show that imposing a bidding cap on the strong player better incentivizes both players. Kirkegaard (2012) shows that it is generally profitable to give the weak player a head start.

Thus the existing auction literature suggests that the underdog should be subsidized in order to improve the competitive balance, which in turn increases effort. Our finding that it is better to subsidize the underdog (the weak player) when the contest is sufficiently accurate (when $r > r^*$), is consistent with the conventional wisdom. However, we also show that this does not carry over when accuracy is low: when the contest is sufficiently "inaccurate" (when $r < r^*$), the strong player should be subsidized.

Fu, Lu and Lu (2012) study the optimal design of R&D contests where the contest designer can split his budget between a prize and efficiency-enhancing (lump-sum) subsidies to the firms. Although we look at similar questions, they adopt a framework where the quality of a firm's product is randomly drawn from a distribution influenced by firm's research capacity and labor input, which is a very different context than the canonical Tullock setting that we consider.[20] Despite the distinctive technical differences, some insights of the two

[19] At the end of Section 1, we discuss situations in which this assumption is reasonable.

[20] Fu and Lu (2012) assume the contest organizer can decrease a player's marginal cost by a given amount by making a lump-sum payment, i.e., this is effectively a change to the technology and it costs the contest designer the same no matter what his effort level is. By contrast, in our model the contest designer can only reimburse a player in proportion to her total cost (i.e., provide a marginal subsidy), so that it costs the contest designer more when the recipient exerts a larger effort.

papers correspond. They find that in the optimally designed contest, the sponsor may favor the strong firm when the innovation process involves substantial difficulty or uncertainty. Analogously, we find that when the contest is sufficiently inaccurate, subsidizing the strong player is preferred.[21]

3 The Tullock Contest Model

3.1 The Two-player Model

There are two risk-neutral players involved in a contest with a single prize V. Player i, $(i = 1, 2)$, has a linear cost function, $c_i(e_i) = c_i \times e_i$, where e_i refers to player i's effort level and $c_i > 0$ is player i's marginal cost of making effort. Denoting $c := c_2/c_1$, assume player 1 is more able than player 2, i.e., $c > 1$. The probability of winning is determined by the following Tullock CSF:[22] In a contest with n contestants, an arbitrary player i wins the prize with probability

$$P_i(e_i, \mathbf{e}_{-i}) = \begin{cases} \dfrac{e_i^r}{\sum_{j=1}^n e_j^r} & \text{if } max\{e_1, ...e_n\} > 0; \\ 1/n & \text{if } max\{e_1, ...e_n\} = 0, \end{cases} \tag{1}$$

where e_i refers to player i's effort level and $\mathbf{e}_{-i} = (e_1, e_2, ...e_{i-1}, e_{i+1}, ..., e_n)$ represents the other $n - 1$ players' effort choices. The parameter r in (1), $r > 0$, which is often referred to as the discriminatory power, can also be interpreted as the accuracy level of the contest. All the parameters, i.e., r, c_1 and c_2, are common knowledge. Each player maximizes his expected utility π_i where $\pi_i = P_i(e_i, \mathbf{e}_{-i})V - c_i e_i$, and has an outside option of zero. Assume that the total effort-maximizing contest designer allocates a fixed amount of resource s to be used to subsidize either player or simply to increase the prize.

Nti (1999) analyzes a model with heterogeneous valuations $(V_1 \geq V_2)$ and homogeneous abilities $(c_1 = c_2 = 1)$. With linear cost functions, there is a close one-to-one relationship between differences in valuations and differences in abilities (i.e., marginal costs). Due to the technical equivalence between heterogeneous valuations and heterogeneous abilities, Wang (2010) obtains the following results from Nti (1999), which we restate as follows:

[21] However, some results differ due to the different settings of the two models. We show that the contest designer always prefers using the resource available on subsidies rather than on increasing the prize, while they show that subsidies should decrease and the prize increase if the innovation process is less challenging.

[22] We present the general version to avoid repetition later.

Lemma 1 *Without subsidies, there exists a unique pure-strategy Nash equilibrium for any* $r \in (0, \bar{r}]$*, where* $\bar{r}$ *satisfies* $c^{\bar{r}} = 1/(\bar{r} - 1)$*, with* $\bar{r}$ *decreasing from 2 to 1 as c increases from 1 to* $+\infty$*. The equilibrium effort levels are:*

$$e_1 = \frac{c_1^r c_2^r rV}{c_1(c_1^r + c_2^r)^2},\ e_2 = \frac{c_1^r c_2^r rV}{c_2(c_1^r + c_2^r)^2},\ TE = \frac{(c_1 + c_2)c_1^{r-1}c_2^{r-1}rV}{(c_1^r + c_2^r)^2}, \tag{2}$$

where $TE := e_1 + e_2$ *denotes total effort.*

Proof. See Appendix.[23] ■

The contest designer has resource s to be allocated to subsidizing players' efforts or to increasing the prize. We focus on the case that when subsidizing a player, a certain proportion of his cost will be covered by the contest designer regardless of who wins the contest. That is, we restrict attention to a linear subsidy scheme. This form of subsidy is referred to as a "marginal subsidy" in this paper. Henceforth in this section we restrict attention to (interior) values of r such that pure-strategy equilibria exist, $r \in (0, \bar{r})$, that is, we rule out r large.[24]

The timing of the model is as follows. Firstly, the contest designer announces her subsidy policy publicly: which player will be subsidized and the level of the marginal subsidy for the recipient, i.e., what percentage of the recipient's cost will be subsidized (or reimbursed). We assume that the resource has no intrinsic value to the contest designer who aims to maximize the total effort. The contest designer can choose either player to subsidize, or if she chooses to subsidize neither player, all the resource will be added directly to the prize. Secondly, given the contest designer's subsidy policy, the two players make their one-shot effort decisions simultaneously in a Tullock contest. Lastly, the contest designer fulfills his commitment as previously announced.

Next, we compare the efficiency of subsidizing one of the two players. Suppose only the strong player is subsidized, so his marginal cost decreases from c_1 to c_1' and he makes an effort e_1' in the new equilibrium. Since the total amount of subsidies must be equal to the

[23] Though Lemma 1 can be derived straightforwardly from Nti (1999), a proof is provided for completeness.

[24] Alcalde and Dahm (2010) and Wang (2010) show that for sufficiently large r such that pure-strategy equilibria do not exist, mixed-strategy equilibria exist. We leave the analysis of such equilibria for future research.

resource that the contest designer possesses,[25] we have

$$s = (c_1 - c_1')e_1', \text{ where } e_1' = \frac{c_1'^r c_2^r rV}{c_1'(c_1'^r + c_2^r)^2}. \tag{3}$$

Alternatively, the contest designer may use s to subsidize the weak player, whose marginal cost of making effort decreases from c_2 to c_2', so that

$$s = (c_2 - c_2')e_2', \text{ where } e_2' = \frac{c_1^r c_2'^r rV}{c_2'(c_1^r + c_2'^r)^2}. \tag{4}$$

It should be stressed that in the above, e_i', $\forall i \in \{1, 2\}$, is player i's *equilibrium* effort given the two players' net of subsidy marginal costs. One issue may be that it is impossible for the designer to deliver on his promises of a marginal subsidy off the equilibrium path if there are no extra resources available. For example, when player 1 is being subsidized, if player 1 exerts more effort than e_1', the contest designer will run out of resources. This concern can be eliminated by simply setting a total subsidy cap s so no more subsidy is provided when e_1' is exceeded. Alternatively we can assume that he has other resources available, but only chooses to allocate s, in equilibrium, to the contest.[26]

We can show that for small s, whichever player i is subsidized, a unique pure-strategy equilibrium continues to exist in which all of s is allocated to player i:

Lemma 2 *Given $r \in (0, \overline{r})$, if the contest designer subsidizes player i, $i \in \{1, 2\}$, then for $s > 0$ sufficiently small there exists a unique pure-strategy equilibrium with c_i' satisfying (3) if $i = 1$ or (4) if $i = 2$. Moreover c_i' is strictly decreasing in s in this range.*

Proof. See Appendix. ■

In view of this result, we can, for s in a neighborhood of 0, write $c_i(s)$ in place of c_i' to be the cost for player i when subsidies s are devoted to player i, and where $c_i(0) = c_i$. By implicit differentiation of (3) and (4):

$$\frac{dc_i(s)}{ds}|_{s=0} = -\left(\frac{1}{e_i}\right), \ i \in \{1, 2\}. \tag{5}$$

[25] We will verify that this is both feasible and desirable below.

[26] If indeed s is a choice variable for the designer, our analysis can be viewed as the solution to the first stage of a two-stage process where the optimal impact of a given level of s is determined, and at the second-stage the level of s is optimized.

Thus, from (2) we get that at $s = 0$,

$$\frac{dc_1(s)}{ds} / \frac{dc_2(s)}{ds} = \frac{e_2}{e_1} = \frac{c_1}{c_2}. \tag{6}$$

To compare impacts on total effort by subsidizing either player, at $s = 0$, write

$$\frac{dTE^S}{ds} / \frac{dTE^W}{ds} = \left(\frac{dTE}{dc_1} / \frac{dTE}{dc_2} \right) \left(\frac{dc_1(s)}{ds} / \frac{dc_2(s)}{ds} \right), \tag{7}$$

where superscripts S and W on TE indicate that we are considering cases where the subsidy is given to the strong or the weak player, respectively. In the Appendix, using (2), (6), we further write (7) as a function of c and r, and obtain the following results.

Proposition 1 *For s sufficiently small,*[27] *(i) in order to maximize total effort,it is better to subsidize the weak player rather than the strong player when $r > r^*$, and the strong player should be subsidized when $0 < r < r^*$ since,*

$$\frac{dTE^S}{ds}|_{s=0} <, > \frac{dTE^W}{ds}|_{s=0} \text{ when } r >, < r^*,$$

where r^* satisfies

$$c^{r^*} - \frac{2r^*(c+1) + (c-1)}{2r^*(c+1) - (c-1)} = 0; \tag{8}$$

(ii) $r^* < 1$ and is decreasing in c.

Proof. See Appendix. ■

An alternative way of increasing total effort is to add the resource s directly to the prize. If the contest designer adds s to the prize, using (2) we have

$$\frac{dTE^P}{ds} = \frac{(c_1 + c_2)c_1^{r-1}c_2^{r-1}r}{(c_1^r + c_2^r)^2}, \tag{9}$$

where superscript P refers to the case when adding s to the prize.

By analyzing $\frac{dTE^S}{ds} / \frac{dTE^P}{ds}$ and $\frac{dTE^W}{ds} / \frac{dTE^P}{ds}$, the following result can be obtained.

Proposition 2 *For s sufficiently small, in order to maximize total effort, subsidizing the strong player is preferred to increasing the prize when $0 < r < 1$; subsidizing the weak player is preferred to increasing the prize when $r \geq 1$.*

[27] When s is sufficiently small, it is sufficient to rank the derivatives $\frac{dTE^X}{ds}$ where $X = S, W, P$, since $TE^{X\prime} \approx TE + s\frac{dTE^X}{ds}$. Here P denotes the case, considered below, where s is used to increase the prize.

Proof. See Appendix. ■

Propositions 1 and 2 imply our main conclusion:

Corollary 1 *In order to maximize total effort, among the three options of using a small amount of resource s (i.e., subsidizing the weak player, subsidizing the strong player or increasing the prize), subsidizing the strong player dominates the other two options when* $0 < r < r^*$*, while subsidizing the weak player dominates the other two options when* $r > r^*$*.*[28]

3.2 Discussion

We discuss the relative benefits of subsidizing either the weak or the strong player, which will give some intuition behind the results of the previous subsection. There is both a direct and an indirect effect of subsidizing a player. First, the direct effect. From (2), the following can be easily derived:

$$\frac{de_1}{dc_1} < 0, \; \frac{de_2}{dc_2} < 0, \tag{10}$$

$$\frac{de_1}{dc_2} < 0, \; \frac{de_2}{dc_1} > 0. \tag{11}$$

Inequalities (10) indicate that subsidizing either player will induce that player to exert more effort.[29] Intuitively: subsidizing a player effectively means he is more able as his marginal cost decreases; thus he will exert more effort in equilibrium. To see whether the weak or the strong player responds more when being subsidized, we have that at $s = 0$:

$$\frac{de_1^S}{ds}/\frac{de_2^W}{ds} = \left(\frac{de_1}{dc_1}/\frac{de_2}{dc_2}\right)\left(\frac{dc_1}{ds}/\frac{dc_2}{ds}\right) = \left(\frac{de_1}{dc_1}/\frac{de_2}{dc_2}\right)\left(\frac{1}{c}\right), \tag{12}$$

where superscripts S and W on e_1 and e_2 indicate that we are considering cases where the subsidy is given to the strong or the weak player, respectively. By analyzing (12), we obtain the following result.

Proposition 3 *(i) It can be shown that*

$$\frac{de_1}{dc_1}/\frac{de_2}{dc_2} > 1$$

[28]Since from (9) $dTE^P/ds > 0$, it follows that the optimal scheme, which is by definition at least as good, also delivers a local increase in aggregate effort and hence it is optimal to use all of s when s is small.

[29]This is the *equilibrium* response of the player who is subsidized.

and this ratio is decreasing in r; (ii) It can be further shown that

$$\frac{de_1^S}{ds}|_{s=0} >,=,< \frac{de_2^W}{ds}|_{s=0} \text{ when } r <,=,> 1.$$

Proof. See Appendix. ■

The fact that the strong player exerts more effort in equilibrium implies that the subsidy goes less far in reducing c_1—this is reflected by the term $(1/c)$ in (12), which was derived in (6): at $s = 0$,

$$\frac{dc_1}{ds} / \frac{dc_2}{ds} = \frac{1}{c}.$$

However by Proposition 3 (i), a given cut in marginal cost has a larger impact on the strong player. Moreover when $r < 1$, this is sufficient to outweigh that there is a smaller reduction in c_1, but not when $r > 1$, which is expressed in Proposition 3 (ii).

Inequalities (11) show the indirect effect. The weak player reduces effort when the strong player is subsidized; by contrast, the strong player exerts more effort when the weak player is subsidized. Intuitively, we can think of the weak player as shrinking from fiercer competition as the strong opponent gets stronger, while the strong player competes more as the weak opponent gets stronger. In sum, the direct effect favors subsidizing the strong player when $r < 1$ and the indirect effect always favors subsidizing the weak player.

When r is sufficiently low (i.e., $0 < r < r^*$), the direct effect of a subsidy to the strong player is large enough, despite the offsetting negative indirect effect on the weak player, to dominate subsidizing the weak player. For $r \geq 1$, the direct effect of subsidizing the weak player is greater (or the same when $r = 1$) from Proposition 3. Since the indirect effect also goes the same way, subsidizing the weak player is preferable. For $r^* < r < 1$, although the direct effect on the strong player is larger, the indirect effects offset this and subsidizing the weak player remains optimal. Moreover, r^* decreasing with c indicates that as the ability difference becomes larger, subsidizing the weak player is more likely to be preferred.

Roughly speaking, the contest designer has to take into account the following two issues when subsidizing a player. First, from the angle of "individual efficiency", the strong player should be subsidized as he is more efficient in exerting effort per se, which is the main reason why the direct effect favors subsidizing the strong player when $r < 1$. Secondly, from the angle of "competitive balance", the weak player should be subsidized as it makes the contest more evenly balanced and thus stimulates competition, which explains why his opponent makes more (less) effort when the weak (strong) player is subsidized.

When the contest is sufficiently noisy ($r < r^*$), the competition between players is not so intense, and the benefit from individual efficiency outweighs the loss from competitive balance, so the strong player should be subsidized; while when the contest is more accurate ($r > r^*$), competition between players is more intense, and the benefit from competitive balance outweighs the loss from individual efficiency, so the weak player should be subsidized.

3.3 With Symmetric Players

In the previous analysis with asymmetric players, we analyzed the two-player case with resource s small. In this subsection, we look at a model with n players when a full range of resource s is allowed, i.e., $s \in (0, +\infty)$, but with symmetric players.

In an n-player Tullock contest model where every player's marginal cost of exerting effort is $\overline{c}$, i.e., $c_1 = c_2 = ... = c_n = \overline{c}$, with CSF given by (1), when $r \leq \overline{r}_n := \frac{n}{n-1}$ a pure-strategy equilibrium can be shown to exist. In the following analysis, we focus on two situations assuming $r \leq \overline{r}_n$: First, the resource s is used as additional prize; second, s is used to provide equal marginal subsidies to all players.

When the resource s is used to supplement the prize, it can be shown that in the pure-strategy equilibrium, each player i's effort level satisfies

$$e_i^P = \left(\frac{n-1}{n^2}\right)\frac{r(V+s)}{\overline{c}}. \tag{13}$$

When the resource s is used to subsidize each player equally, in equilibrium each player i receives s/n as subsidies:

$$\frac{s}{n} = (\overline{c} - \overline{c}')e_i', \text{ where } e_i' = \left(\frac{n-1}{n^2}\right)\frac{rV}{\overline{c}'}. \tag{14}$$

Thus

$$e_i' = \frac{(n-1)rV + ns}{n^2\overline{c}}. \tag{15}$$

As $\overline{r}_n = \frac{n}{n-1}$, (15) can be rewritten as

$$e_i' = \left(\frac{n-1}{n^2}\right)\frac{rV + \overline{r}_n s}{\overline{c}}. \tag{16}$$

When a pure-strategy equilibrium exists, $r \le \overline{r}_n$, and clearly $e_i^P \le e_i'$ as RHS of (13) $\le$ RHS of (16): allocating s to the prize is strictly dominated by subsidizing players (evenly) unless $r = \overline{r}_n$.[30,31] It is straightforward to show that any split allocation of the resource s between prize and subsidy is dominated by allocating s entirely as subsidy. We summarize the above result in the following proposition:

Proposition 4 *With n symmetric players, for r such that a pure strategy equilibrium exists ($r \le \overline{r}_n$), allocating any of the resource s as an additional prize is dominated by using s entirely as an equal subsidy for all players.*

The above result also implies that, with symmetric players, if the original prize V can be partially allocated as subsidies, then the contest designer has an incentive to reallocate resources from the prize to subsidies to the maximum extent possible. In fact a player's expected equilibrium payoff would approach zero as $V \to 0$ and each player is almost fully subsidized, with effort being so large that rent is almost fully dissipated by the contest designer. In practice however the contest prize is not usually completely divisible and cannot be fully or even partially reallocated as subsidies, as for example with the contract offered by the DoD, or higher exam grades offered by a school.

4 The Lottery Contest Model

In a setting of a general Tullock contest,[32] analytical tractability restricted our preceding analysis to the two-player case with resource s being sufficiently small (and the n-player case with symmetric players). In this section we look at lottery contests (i.e., Tullock contests with $r = 1$) which are the most widely studied special cases of Tullock contests. This setting allows us to analyze the following two situations: the n-player model ($n \ge 2$) and the two-player model with $s \in (0, +\infty)$.

[30] When $r = \overline{r}_n$, in the pure-strategy equilibrium each player's expected payoff is zero regardless of the value of the prize, i.e., the rent is fully dissipated.

[31] This analysis can straightforwardly accommodate nonlinear effort costs. Suppose the cost function takes the form $\overline{c}e_i^{\alpha}$, $\alpha > 0$. Define $\widetilde{r} = r/\alpha$. Note that pure-strategy equilibria exist when $\widetilde{r} \le \overline{r}_n$. We would have that allocating s to the prize is strictly dominated when $\widetilde{r} < \overline{r}_n$, and they are equivalent when $\widetilde{r} = \overline{r}_n$ in which case each player's expected payoff is zero.

[32] As mentioned earlier, here we add "general" in the sense that r can take any value provided a pure-strategy equilibrium exists.

4.1 The N-player Lottery Contest Model

We consider a model with n risk-neutral contestants in a contest with a single prize V, where the CSF is the Tullock CSF (1) with $r = 1$. Assume player i's marginal cost of making effort e_i is c_i. The contestants are asymmetric in the sense that $0 < c_1 \leq c_2 \leq ... \leq c_n$ and $c_1 < c_n$. Player 1 (whose marginal cost is c_1) is the most "able" player and player n (whose marginal cost is c_n) is the least able. Assume we start with an equilibrium with no subsidy and all players participating in the contest with strictly positive efforts.

Player i's expected profit is $\pi_i = P_i(e_i, \mathbf{e}_{-i})V - c_i e_i$, where $P_i(e_i, \mathbf{e}_{-i})$ is given by (1) with $r = 1$. The first order condition for player i is

$$\frac{d\pi_i}{de_i} = \frac{V \sum_{j \neq i} e_j}{(e_i + \sum_{j \neq i} e_j)^2} - c_i = 0. \tag{17}$$

It can be verified that the second-order condition for $d^2\pi_i/de_i^2$ is satisfied for any $n \geq 2$.

Lemma 3 *(i)* [33] *In the n-player model, total effort is*

$$TE \equiv \textstyle\sum_{i=1}^n e_i = \frac{V(n-1)}{\sum_{i=1}^n c_i}; \tag{18}$$

(ii) For any two players, player p *and player* q*, with* $c_p < c_q$*, it must be the case that* $e_p > e_q$*, i.e., the more able the player is, the more effort he exerts in equilibrium.*

Proof. See Appendix. ■

As in the two-player model, suppose the contest designer has a small amount of resource s which can be used to subsidize any player or increase the prize directly. For any two players p and q with $c_p < c_q$ in the initial equilibrium with no subsidy, their effort levels are $e_q < e_p$. When player p is subsidized, his marginal cost decreases from c_p to c'_p, and he exerts new effort level e'_p. The total amount of subsidies must be equal to the resource that the contest designer has available, i.e.,

$$s = (c_p - c'_p)e'_p = \Delta c_p e'_p.$$

Likewise when player q is subsidized,

$$s = (c_q - c'_q)e'_q = \Delta c_q e'_q.$$

[33] The result in (i) has been derived in Stein (2002) and Ritz (2008). The uniqueness of equilibrium in asymmetric contests is established by Matros (2006).

Thus $\Delta c_p e'_p = s = \Delta c_q e'_q$, which implies that $\Delta c_p < \Delta c_q$ as for small s,

$$e'_p \approx e_p > e_q \approx e'_q.$$

From (18), we can see that in equilibrium the total effort only depends on the sum of the marginal costs, i.e., $\sum_{i=1}^{n} c_i$. Therefore, regardless of who is being subsidized, the contest designer only cares about the change of the recipient's marginal cost. Thus, subsidizing a less able player (player q) yields a larger total effort than subsidizing a more able player (player p) as $\Delta c_p < \Delta c_q$. Hence it is optimal to subsidize the weakest player to maximize the total effort.

In the Appendix, we also show that subsidizing the weakest player is more effective in increasing total effort than increasing the prize. We summarize the findings in the following proposition.[34]

Proposition 5 *For sufficiently small s, to maximize total effort: (i) subsidizing the weakest player (among players who are willing to make strictly positive efforts) is more efficient than subsidizing any other player; (ii) subsidizing the weakest player is also more efficient than adding the resource to the prize.*

Proof. See Appendix for the proof of *(ii)*. ■

The above result is in line with our previous finding in a general Tullock contest model: when $r = 1$, subsidizing the weak player is more effective in increasing total effort than subsidizing the strong player or increasing the prize.

4.2 The Two-player Lottery Contest Model with $s \in (0, +\infty)$

In this model, we allow for s large and divisible, so that, for instance, it is possible to subsidize both players simultaneously.

Assume that the contest designer uses s_1 and s_2 to simultaneously subsidize player 1 and player 2 respectively, and also adds s_P to the prize at the same time, where $s_1 + s_2 + s_P \leq s$. We seek to find the optimal allocation of s_1, s_2 and s_P that maximizes total effort.

[34] The argument that all of s should be expended when s is small is as discussed above in the general two-player case.

After being subsidized simultaneously, player 1's marginal cost decreases from c_1 to c_1', and player 2's marginal cost decreases from c_2 to c_2'. Let e_1' and e_2' denote player 1 and 2's effort levels after being subsidized respectively and let $TE' = e_1' + e_2'$. Thus, using (2) the following must hold:

$$s_1 = (c_1 - c_1')e_1', \ \ s_2 = (c_2 - c_2')e_2', \tag{19}$$

where

$$e_1' = \frac{c_2'(V + s_P)}{(c_1' + c_2')^2}, \ e_2' = \frac{c_1'(V + s_P)}{(c_1' + c_2')^2}. \tag{20}$$

From (19) and (20), we get

$$s_1 = (c_1 - c_1')\frac{c_2'(V + s_P)}{(c_1' + c_2')^2}, \tag{21}$$

$$s_2 = (c_2 - c_2')\frac{c_1'(V + s_P)}{(c_1' + c_2')^2}. \tag{22}$$

It is straightforward (but tedious) to establish that for given values of c_1, c_2, V and s, for every possible allocation $(s_1, s_2, s_P) \in \mathbb{R}_+^3$ of the resource s where $s_1 + s_2 + s_P \leq s$, there exists a pair (c_1', c_2') that is the unique solution to the system of equations (21) and (22).[35] This (c_1', c_2') then corresponds to the unique pure-strategy equilibrium associated with the allocation (s_1, s_2, s_P).[36] Defining

$$\Delta c_1 := c_1 - c_1' \text{ and } \Delta c_2 := c_2 - c_2',$$

then by (19) and (20),

$$s_1 + s_2 = \Delta c_1 e_1' + \Delta c_2 e_2'; \tag{23}$$

$$TE' = \frac{V + s_P}{c_1 + c_2 - \Delta c_1 - \Delta c_2}. \tag{24}$$

It is straightforward to see that it is optimal to use the entire resource to maximize total effort. Suppose otherwise, i.e., that total effort is maximized by using a smaller resource $\hat{s} < s$. But then increasing the prize will further increase effort, so using only $\hat{s}$ cannot be optimal,[37] and it follows that all of s will be used.

[35] It can be checked that the Jacobian determinant of the system of equations (21) and (22) with respect to c_1' and c_2' is always non-zero for positive subsidies so this solution is also differentiable in subsidies by the Implicit Function Theorem.

[36] By the logic of Footnote 33.

[37] Consider, starting from $s_1 + s_2 + s_P = \hat{s} < s$ a small increase in the prize holding c_1' and c_2' constant. From (20) e_1' and e_2' are continuous increasing functions of s_P so will increase by small amounts, implying that effort increases, but the change in $s_1 + s_2$ is also small from (19) so total spending will remain below s and the change is feasible. That total effort increases contradicts the assumed initial optimality.

In the Appendix, we establish the following result.

Proposition 6 *In order to maximize total effort, (i) when $s \leq \bar{s}$, where*

$$\bar{s} = \frac{2(c-1)V}{(3+c)^2}, \tag{25}$$

it is optimal to use all the resource s to subsidize the weak player; (ii) when $s > \bar{s}$, it is optimal to use all the resource to subsidize both players simultaneously. Increasing the prize is always dominated.

Proof. See Appendix. ■

Our analysis in section 2 showed that for the case $r = 1$, when s is sufficiently small, subsidizing the weak player always dominates. Proposition 6 (i) shows that this result continues to hold when the resource is larger so long as $s \leq \bar{s}$.[38]

In our previous analysis with s being sufficiently small, we only needed to compare the marginal effects of the three options (i.e., subsidizing the strong/weak player or adding to the prize), and devote all the resource to the option that induces the largest marginal increase in total effort. But for larger resource it is possible that using two or three options simultaneously may dominate using a single option. Proposition 6 (ii) indicates that for higher values of s ($s > \bar{s}$), the strong player is also subsidized, but the prize is never increased. Subsidizing the strong player becomes optimal as subsidizing just the weaker player would create a competitive imbalance.

5 Concluding Comments

This paper has examined a situation where the contest designer has a limited amount of resource s which can be used to provide marginal subsidies or to directly increase the prize.

In a two-player Tullock model, we show that it is optimal to subsidize the strong (weak) player when the contest is noisy (accurate). Intuitively, from the point of view of "individual efficiency", the strong player should be subsidized as he is more efficient in exerting effort. However, considering "competitive balance", the weak player should be subsidized as it will make the contest more evenly balanced, which in turn stimulates competition and effort.

[38] For example, our numerical analysis shows that when c ($= c_2/c_1$) $= 3$, $\bar{s} = 0.11V$.

When the contest is noisy ($r < r^*$), competition is not intense and "individual efficiency" outweighs "competitive balance" and the strong player should be subsidized. The reverse occurs when the contest is fairly accurate ($r > r^*$), i.e., the weak player should be subsidized.

In a n-player lottery contest (i.e., Tullock contest with $r = 1$), we show that, to maximize total effort, it is optimal to subsidize the weakest player, which is consistent with our previous findings in a general Tullock contest. Moreover, in a two-player lottery contest, we characterize the optimal scheme for all possible values of $s \in (0, +\infty)$: as our results in a general Tullock contest imply, when s is small ($s < \bar{s}$), it is optimal to only subsidize the weak player; while when s is large ($s > \bar{s}$), both players should be subsidized simultaneously; but again, increasing the prize is always dominated.

Although we considered lottery contests for a general case with n players and $s \in (0, +\infty)$, our analysis of a general Tullock contest was restricted to the two-player case with resource s being sufficiently small. Despite the seeming technical difficulties, an analysis in a general Tullock contest with more than two players and a full range of resource where $s \in (0, +\infty)$, and further analysis of the optimal subsidy scheme within other model settings, merit further study.

6 Appendix

6.1 Proof of Lemma 1

The expected revenue for contestants 1 and 2 are:

$$L_1 = \frac{e_1^r V}{e_1^r + e_2^r} - c_1 e_1; \ L_2 = \frac{e_2^r V}{e_1^r + e_2^r} - c_2 e_2. \tag{26}$$

Assume there exists a pure-strategy equilibrium where contestants 1 and 2 make efforts e_1^* and e_2^* respectively. First order conditions require

$$\frac{\partial L_1}{\partial e_1}|_{e_1=e_1^*, e_2=e_2^*} = 0; \frac{\partial L_2}{\partial e_2}|_{e_1=e_1^*, e_2=e_2^*} = 0,$$

which implies

$$e_1^{*r} e_2^{*r} r V = c_1 e_1^* (e_1^{*r} + e_2^{*r})^2; \ e_1^{*r} e_2^{*r} r V = c_2 e_2^* (e_1^{*r} + e_2^{*r})^2. \tag{27}$$

Solving (27), we have

$$e_1^* = \frac{c_1^r c_2^r r V}{c_1 (c_1^r + c_2^r)^2}, \ e_2^* = \frac{c_1^r c_2^r r V}{c_2 (c_1^r + c_2^r)^2}. \tag{28}$$

To ensure the existence of a pure-strategy equilibrium, all contestants' participation constraints must hold; substituting (28) into (26), we require:

$$L_1|_{e_1=e_1^*;\ e_2=e_2^*} = \frac{(1-c^r(r-1))V}{(1+c^r)^2} \geq 0;$$

$$L_2|_{e_1=e_1^*;\ e_2=e_2^*} = \frac{c^r(1+c^r-r)V}{(1+c^r)^2} \geq 0.$$

Thus,

$$L_1|_{e_1=e_1^*;\ e_2=e_2^*} \geq 0 \Leftrightarrow c^r \geq r-1; \tag{29}$$

$$L_2|_{e_1=e_1^*;\ e_2=e_2^*} \geq 0 \Leftrightarrow c^r(r-1) \leq 1. \tag{30}$$

When $r \leq 1$, we have $c^r(r-1) \leq 1$ and $c^r \geq r-1$. So when $r \leq 1$, the equilibrium always exists. When $r > 1$, a necessary and sufficient condition for the equilibrium to exist, by (29) and (30), is:

$$r-1 \leq c^r \leq \frac{1}{r-1}. \tag{31}$$

A necessary condition for (31) is $r \leq 2$, otherwise $r-1 > 1/(r-1)$. When $r \leq 2$, we always have $r-1 \leq c^r$. So we only need focus on $c^r \leq 1/(r-1)$. When $r > 1$, c^r is increasing in r and $1/(r-1)$ is decreasing in r, so we need $r \leq \overline{r}$, where $\overline{r}$ satisfies $c^{\overline{r}} = 1/(\overline{r}-1)$. In summary, to ensure players' participation constraints hold, we need $0 \leq r \leq \overline{r}$, where $\overline{r}$ satisfies $c^{\overline{r}} = 1/(\overline{r}-1)$. It is easily checked that when c increases from 1 to $+\infty$, $\overline{r}$ decreases from 2 to 1.

Next we want to show that the second order conditions hold at equilibrium when $e_1 = e_1^*$, $e_2 = e_2^*$. For player 1,

$$\frac{d^2L_1}{de_1^2}|_{e_1=e_1^*;\ e_2=e_2^*} = -\frac{e_2^{*r}e_1^{*-2+r}rV}{(e_1^{*r}+e_2^{*r})^3}[(r+1)e_1^{*r}-(r-1)e_2^{*r}]$$

Notice that (27) implies that $e_1^{*r} = (c_2/c_1)e_2^{*r} = ce_2^{*r}$, so

$$\begin{aligned}\frac{d^2L_1}{de_1^2}|_{e_1=e_1^*;\ e_2=e_2^*} &= -\frac{e_2^{*r}rVe_1^{*-2+r}}{(e_1^{*r}+e_2^{*r})^3}[(r+1)(ce_2^*)^r-(r-1)e_2^{*r}] \\ &= -\frac{rVe_1^{*-2+r}e_2^{*2r}}{(e_1^{*r}+e_2^{*r})^3}[(r+1)c^r-(r-1)].\end{aligned}$$

As $[(r+1)c^r-(r-1)] > 1+r+1-r > 0$, we have

$$\frac{d^2L_2}{de_2^2}|_{e_1=e_1^*;\ e_2=e_2^*} < 0.$$

For player 2,

$$\frac{d^2L_2}{de_2^2}|_{e_1=e_1^*;\ e_2=e_2^*} = \frac{e_1^{*r}e_2^{*-2+2r}rV}{(e_1^{*r}+e_2^{*r})^3}[c^r(r-1))-(r+1)].$$

As (30) implies $c^r(r-1) \leq 1$, $c^r(r-1)-(r+1) < 0$, and we have

$$\frac{d^2L_2}{de_2^2}|_{e_1=e_1^*;\ e_2=e_2^*} < 0.$$

6.2 Proof of Lemma 2

Suppose the contest designer subsidizes the strong player, and consider the following (see (3)):

$$s = g(c_1') := \frac{(c_1 - c_1')c_1'^{r-1}c_2^r rV}{(c_1'^r + c_2^r)^2}. \tag{32}$$

From (32), we get

$$\frac{dg(c_1')}{dc_1'} = \frac{c_1'^{r-2}c_2^r r[c_1c_2^r(r-1) - c_1'r(c_2^r - c_1'^r) - c_1c_1'^r(r+1)]V}{(c_1'^r + c_2^r)^3}. \tag{33}$$

If $r \leq 1$, $dg(c_1')/dc_1' < 0$ as $c_1c_2^r(r-1) \leq 0$ and $-c_1'r(c_2^r - c_1'^r) - c_1c_1'^r(r+1) < 0$. If $r > 1$, then as by assumption $r < \overline{r}$, and by definition $c^{\overline{r}} = 1/(\overline{r} - 1)$, we get

$$c^r < \frac{1}{r-1},$$

which by $c := c_2/c_1$ further implies $c_2^r(r-1) < c_1^r$; thus we get

$$\begin{aligned} \frac{dg(c_1')}{dc_1'}\Big|_{c_1'=c_1} &< \frac{c_1^{r-2}c_2^r r[c_1c_1^r - c_1r(c_2^r - c_1^r) - c_1c_1^r(r+1)]V}{(c_1^r + c_2^r)^3} \\ &= \frac{c_1^{r-2}c_2^r r[-c_1r(c_2^r - c_1^r) - c_1c_1^r r]V}{(c_1^r + c_2^r)^3} < 0. \end{aligned}$$

Thus by the implicit function theorem for s in a neighbourhood of 0 there exists a unique c_1' that satisfies (32) (as a differentiable function of s), and in particular for $s > 0$ sufficiently small and defining $c' := c_2/c_1'$, c' is arbitrarily close to c, so we can choose s small enough so that $\overline{r}'$ which solves $(c')^{\overline{r}'} = 1/(\overline{r}' - 1)$ is close to $\overline{r}$ (the upper bound without subsidies by Lemma 1), and in particular $r \in (0, \overline{r}']$. By Lemma 1 this guarantees that a pure-strategy equilibrium exists for all s small enough and $dg(c_1)/dc_1' < 0$ implies that c_1' is decreasing in s.

Likewise when the contest designer subsidizes the weak player, from (4),

$$s = g(c_2') := \frac{(c_2 - c_2')c_1^r c_2'^{r-1} rV}{(c_1^r + c_2'^r)^2}. \tag{34}$$

From (34) we get

$$\frac{dg(c_2')}{dc_2'} = -\frac{c_2'^{r-2}c_1^r r[c_2'^r(c_2 + c_2r - c_2'r) + c_1^r(c_2 - c_2r + c_2'r)]V}{(c_1'^r + c_2^r)^3}. \tag{35}$$

If $r \leq 1$, then $dg(c_2')/dc_2' < 0$ as $c_2'^r(c_2 + c_2r - c_2'r) > 0$ and $c_2 - c_2r + c_2'r > 0$. If $r > 1$, as shown above, $c_2^r(r-1) < c_1^r$; thus we get

$$\begin{aligned} \frac{dg(c_2')}{dc_2'}\Big|_{c_2'=c_2} &< -\frac{c_2^{r-2}c_1^r r[c_2^r(c_2 + c_2r - c_2r) + c_2^r(r-1)(c_2 - c_2r + c_2r)]V}{(c_1^r + c_2^r)^3} \\ &= -\frac{c_2^{2r-1}c_1^r r^2 V}{(c_1^r + c_2^r)^3} < 0. \end{aligned}$$

Then by analogous reasoning to the above case with the strong player being subsidized, for s in a neighbourhood of 0 there exists a unique c_2' that satisfies (34) (as a differentiable decreasing function of s), and such that a pure-strategy equilibrium exists.

6.3 Proof of Proposition 1

From the expression for TE in (2), we get

$$\frac{dTE}{dc_1} = \frac{c_1^{r-2}c_2^{r-1}\{c_2^r[c_2(r-1)+c_1 r] - c_1^r[c_2+(c_1+c_2)r]\}rV}{(c_1^r+c_2^r)^3}, \tag{36}$$

$$\frac{dTE}{dc_2} = \frac{c_1^{r-1}c_2^{r-2}\{c_1^r[c_1(r-1)+c_2 r] - c_2^r[c_1+(c_1+c_2)r]\}rV}{(c_1^r+c_2^r)^3}. \tag{37}$$

Substituting (6), (36) and (37) into (7), we derive that at $s=0$,

$$\frac{dTE^S}{ds}/\frac{dTE^W}{ds} = \frac{[c+(1+c)r] - c^r[c(r-1)+r]}{c^r[1+(1+c)r] - [(r-1)+cr]}. \tag{38}$$

From (38), it can be derived that

$$\frac{dTE^S}{ds}|_{s=0} >,< \frac{dTE^W}{ds}|_{s=0}$$

if and only if $f(r) >, < 0$, where

$$\begin{aligned} f(r) &:= [c+(1+c)r] - c^r[c(r-1)+r] - c^r[1+(1+c)r] + [(r-1)+cr] \\ &= 2r(c+1)+(c-1) - c^r[2r(c+1)-(c-1)]. \end{aligned} \tag{39}$$

It is straightforward to see that $f(r) > 0$ when $2r(c+1)-(c-1) \leq 0$, i.e., $f(r) > 0$ when $r \leq (c-1)/2(c+1)$. It can also be proved that $f(r) < 0$ when $r \geq 1$. This is because at the point $r=1$, $f(r) = 1-c^2 < 0$ and

$$\frac{df(r)}{dr} = -\left[2(1+c)(c^r-1) + c^r(1-c+2(1+c)r)\log(c)\right] < 0. \tag{40}$$

Therefore, we have shown that $f(r) > 0$ when $r \leq (c-1)/2(c+1)$ and $f(r) < 0$ when $r \geq 1$. What is $f(r)$ when $(c-1)/2(c+1) < r < 1$? Using (39), we can show that $f = 2(c-1) > 0$ when $r = (c-1)/2(c+1)$ and $f < 0$ when $r = 1$. Using (40), we get that $df(r)/dr < 0$ when $(c-1)/2(c+1) < r < 1$, i.e., $f(r)$ is strictly decreasing when r increases. Thus, we conclude that there must exist a r^* where $f(r^*) = 0$ and $(c-1)/2(c+1) < r^* < 1$. It follows that when $r > r^*$, $f(r) < 0$; and when $r < r^*$, $f(r) > 0$.

Because of $f(c, r^*) = 0$ (adding c as an explicit argument to f) we have

$$\frac{dr^*}{dc} = -\left(\frac{\partial f}{\partial c}/\frac{\partial f}{\partial r^*}\right). \tag{41}$$

It is simple to derive that

$$\frac{\partial f}{\partial r^*} = -\left[2(1+c)(c^{r^*}-1) + c^{r^*}(1-c+2(1+c)r^*)\log(c)\right] < 0 \tag{42}$$

and

$$\frac{\partial f}{\partial c} = 2r^* + 1 - c^{r^*-1}[2(1+c)r^{*2} + (1+c)r^* - c]. \tag{43}$$

Since $f(r^*) = 0$, from (39) we have

$$c^{r^*} = \frac{2r^*(c+1) + (c-1)}{2r^*(c+1) - (c-1)}. \tag{44}$$

Substituting (44) into (43), we have

$$\frac{\partial f}{\partial c} = -r^* \left\{ \frac{4r^{*2}(c+1)^2 - (c-1)^2 - 8c}{c[(1-c) + 2(1+c)r^*]} \right\}.$$

Since $(1-c) + 2(1+c)r^* > 0$, $\partial f/\partial c < 0$ if and only if

$$4r^{*2}(c+1)^2 - (c-1)^2 - 8c > 0,$$

$$\text{i.e., } r^* > \frac{\sqrt{8c + (c-1)^2}}{2(c+1)} = \widetilde{r}. \tag{45}$$

Substituting $\widetilde{r}$ into (39), we can show that $f(\widetilde{r}) > 0$ for all $c > 0$. It must be the case that $r^* > \widetilde{r}$ since $f(\widetilde{r}) > f(r^*) = 0$ and $df/dr < 0$. Thus, $\partial f/\partial c < 0$. Therefore,

$$\frac{dr^*}{dc} = -\left(\frac{\partial f}{\partial c} / \frac{\partial f}{\partial r^*}\right) < 0.$$

We have shown that r^* decreases when c increases. By numerical analysis, we can derive that $r^* \approx 0.708$ when $c \to 1$ and $r^* \approx 0.5$ when $c \to +\infty$.

6.4 Proof of Proposition 2

If the contest designer subsidizes the weak player, using (5) we have

$$\frac{dTE^W}{ds}|_{s=0} = \frac{dTE}{dc_2}\left(\frac{dc_2}{ds}\right) = \frac{dTE}{dc_2}\left(-\frac{1}{e_2}\right). \tag{46}$$

Substituting the expression of e_2 in (2) and (37) into (46),

$$\frac{dTE^W}{ds}|_{s=0} = \frac{c_2^r[c_1 + (c_1+c_2)r] - c_1^r[c_1(r-1) + c_2 r]}{c_1 c_2 (c_1^r + c_2^r)}. \tag{47}$$

Using (9) and (47), we get at $s = 0$,

$$\frac{dTE^W}{ds} / \frac{dTE^P}{ds} = \left[1 + \frac{c^r + 1 - (1+c)r}{c^r(1+c)r}\right](1 + c^r). \tag{48}$$

Moreover, to compare the effect of adding s to the prize with that of subsidizing the strong player, we have:

$$\frac{dTE^S}{ds} / \frac{dTE^P}{ds} = \left(\frac{dTE^S}{ds} / \frac{dTE^W}{ds} \right) \left(\frac{dTE^W}{ds} / \frac{dTE^P}{ds} \right). \tag{49}$$

Substituting (38) and (48) into (49), we derive that at $s = 0$,

$$\frac{dTE^S}{ds} / \frac{dTE^P}{ds} = \left\{ 1 + \frac{c - c^r[r - c(1-r)]}{(1+c)r} \right\} (1 + c^{-r}). \tag{50}$$

From (50), we can see that when $0 < r \leq 1$,

$$c - c^r[r - c(1-r)] = c - c^r r + c^{r+1}(1-r) > 0$$

since $c > c^r r$ and $c^{r+1}(1-r) > 0$. Thus, at $s = 0$,

$$\frac{dTE^S}{ds} > \frac{dTE^P}{ds} \text{ when } 0 < r \leq 1.$$

When $r \geq 1$, from (48), at $s = 0$,

$$\begin{aligned} \frac{dTE^W}{ds} / \frac{dTE^P}{ds} &= \left[1 + \frac{c^r + 1 - (1+c)r}{c^r(1+c)r} \right] (1 + c^r) \\ &> 2 \left[1 + \frac{c + 1 - (1+c)r}{c^r(1+c)r} \right] \\ &= 2 \left[\frac{c^r r - (r-1)}{c^r r} \right]. \end{aligned}$$

Thus, $\frac{dTE^W}{ds} / \frac{dTE^P}{ds} > 1$ requires that

$$2 \left[\frac{c^r r - (r-1)}{c^r r} \right] \geq 1,$$

i.e., $c^r r \geq 2(r-1)$. It can be shown that when a pure-strategy equilibrium exists, $r \leq 2$, $2(r-1) \leq 1 < c^r r$ when $r \geq 1$, finally we have $c^r r \geq 2(r-1)$.

6.5 Proof of Proposition 3

From (2), we can show that

$$\begin{aligned} \frac{de_1}{dc_1} &= -\frac{c_1^{r-2} c_2^r [c_1^r(1+r) + c_2^r(1-r)] rV}{(c_1^r + c_2^r)^3}; \\ \frac{de_2}{dc_2} &= -\frac{c_1^r c_2^{r-2} [c_1^r(1-r) + c_2^r(1+r)] rV}{(c_1^r + c_2^r)^3}. \end{aligned}$$

Then

$$\frac{de_1}{dc_1}/\frac{de_2}{dc_2} = \frac{c_2^2[c_1^r(1+r)+c_2^r(1-r)]}{c_1^2[c_1^r(1-r)+c_2^r(1+r)]}$$
$$= \frac{c^2[(1+r)+c^r(1-r)]}{[(1-r)+c^r(1+r)]}. \tag{51}$$

Thus

$$\frac{d(\frac{de_1}{dc_1}/\frac{de_2}{dc_2})}{dr} = \frac{-2c^2(c^{2r}-1+2c^r r\log c)}{[(1-r)+c^r(1+r)]^2} < 0;$$

From (12) and (51), we derive that at $s=0$,

$$\frac{de_1^S}{ds}/\frac{de_2^W}{ds} = \frac{c[(1+r)+c^r(1-r)]}{[(1-r)+c^r(1+r)]}.$$

It follows that

$$\frac{de_1^S}{ds}|_{s=0} >,=,< \frac{de_2^W}{ds}|_{s=0}$$

when

$$(c^{r+1}-1)(1-r)-(c^r-c)(1+r) >,=,< 0.$$

Notice that when $r <,=,> 1$, both $(1-r) >,=,< 0$ and $(c^r-c) <,=,> 0$, therefore $(c^{r+1}-1)(1-r)-(c^r-c)(1+r) >,=,< 0$.

6.6 Proof of Lemma 3

a) In equilibrium, each player's effort satisfies (17). By summing (17) over all players, we get:

$$\textstyle\sum_{i=1}^n \frac{d\pi_i}{de_i} = \frac{\sum_{i=1}^n \left\{\sum_{j\neq i} e_j\right\}}{\left(e_i+\sum_{j\neq i} e_j\right)^2}V - \sum_{i=1}^n c_i = 0. \tag{52}$$

Since

$$\textstyle\sum_{i=1}^n(\sum_{j\neq i} e_j) = \sum_{i=1}^n\{(\sum_{i=1}^n e_i)-e_i\} = (n-1)\sum_{i=1}^n e_i, \tag{53}$$

(52) becomes

$$\textstyle\sum_{i=1}^n \frac{d\pi_i}{de_i} = \frac{V(n-1)}{\sum_{i=1}^n e_i} - \sum_{i=1}^n c_i = 0.$$

It follows that

$$TE = \textstyle\sum_{i=1}^n e_i = \frac{V(n-1)}{\sum_{i=1}^n c_i}.$$

b) For any two players with $c_p < c_q$, by using (17) we can write:

$$\frac{d\pi_p}{de_p} = \frac{V(\sum_{i=1}^n e_i - e_p)}{(\sum_{i=1}^n e_i)^2} - c_p = 0; \tag{54}$$
$$\frac{d\pi_q}{de_q} = \frac{V(\sum_{i=1}^n e_i - e_q)}{(\sum_{i=1}^n e_i)^2} - c_q = 0. \tag{55}$$

Thus it follows that

$$\frac{\sum_{i=1}^{n} e_i - e_p}{\sum_{i=1}^{n} e_i - e_q} = \frac{c_p}{c_q} < 1, \tag{56}$$

so that $e_p > e_q$.

6.7 Proof of Proposition 5 (ii)

If the contest designer adds s to the prize in place of subsidizing the weak player, from (18) we have

$$\frac{dTE^P}{ds} = \frac{dTE}{dV} = \frac{n-1}{\sum_{i=1}^{n} c_i}. \tag{57}$$

If the contest designer subsidizes the weakest player, i.e., player n, then

$$\frac{dTE^W}{ds}|_{s=0} = \frac{dTE^W}{dc_n}\frac{dc_n}{ds} = \frac{dTE^W}{dc_n}\left(-\frac{1}{e_n}\right). \tag{58}$$

From (18), we derive

$$\frac{dTE^W}{dc_n} = -\frac{n-1}{\left(\sum_{i=1}^{n} c_i\right)^2}. \tag{59}$$

Because player n is the weakest player who exerts the smallest effort among the players who are willing to enter the contest actively,

$$e_n \leq \frac{TE}{n} = \frac{n-1}{n\sum_{i=1}^{n} c_i}. \tag{60}$$

Substituting (59) and (60) into (58), we get

$$\frac{dTE^W}{ds}|_{s=0} \geq \frac{n}{\sum_{i=1}^{n} c_i} > \frac{n-1}{\sum_{i=1}^{n} c_i},$$

i.e., at $s = 0$,

$$\frac{dTE^W}{ds} > \frac{dTE^P}{ds}.$$

6.8 Proof of Proposition 6

A. We start by assuming that all of s is used on subsidies, i.e., $s_P = 0$. Suppose the optimal total effort level under this restriction is $\overline{TE}$. Consider the dual problem of choosing s_1 and s_2 to minimize the total resource used, $\widetilde{s} = s_1 + s_2$, subject to total effort being $\overline{TE}$. This will lead to the same optimal values for s_1 and s_2.[39]

[39] To be precise, any solution to the dual problem must also be a solution to the primal problem. Since the entire budget must be used up (see footnote 40 below) it will follow that the solution to the primal is unique. We use $\widetilde{s}$ to represent total resource, which is a endogenous variable in this case, to distinguish it from s, which we treat as fixed.

Given $s_P = 0$, using (19), (20) and (24),

$$s_1 = \Delta c_1 e_1' = \frac{\Delta c_1(c_2 - \Delta c_2)V}{(c_1 + c_2 - \Delta c_1 - \Delta c_2)^2} = \Delta c_1(c_2 - \Delta c_2)\frac{\overline{TE}^2}{V},$$
$$s_2 = \Delta c_2 e_2' = \frac{\Delta c_2(c_1 - \Delta c_1)V}{(c_1 + c_2 - \Delta c_1 - \Delta c_2)^2} = \Delta c_2(c_1 - \Delta c_1)\frac{\overline{TE}^2}{V}.$$

Then

$$\widetilde{s} = [\Delta c_1(c_2 - \Delta c_2) + \Delta c_2(c_1 - \Delta c_1)]\frac{\overline{TE}^2}{V}. \tag{61}$$

From (24), given that total effort is fixed at $\overline{TE}$, $\Delta c_1 + \Delta c_2$ must be fixed at some $\overline{\Delta c}$:

$$\overline{\Delta c} = \Delta c_1 + \Delta c_2. \tag{62}$$

Writing $\Delta c_2 = \overline{\Delta c} - \Delta c_1$ and substituting into (61) we have[40]

$$\widetilde{s} = [\Delta c_1(c_2 - \overline{\Delta c} + \Delta c_1) + (\overline{\Delta c} - \Delta c_1)(c_1 - \Delta c_1)]\frac{\overline{TE}^2}{V}. \tag{63}$$

The contest designer chooses Δc_1 in (63) to minimize $\widetilde{s}$. Differentiating with respect to Δc_1 yields

$$(c_2 - c_1 - 2\overline{\Delta c} + 4\Delta c_1)\frac{\overline{TE}^2}{V} =: K(\Delta c_1)\frac{\overline{TE}^2}{V}. \tag{64}$$

When $\overline{\Delta c} \leq \frac{c_2 - c_1}{2}$, (64) is non-negative since

$$K(\Delta c_1) \geq [c_2 - c_1 - 2(\frac{c_2 - c_1}{2}) + 4\Delta c_1] = 4\Delta c_1 \geq 0;$$

so, in order to minimize (63), it is optimal to set Δc_1 to its minimal value, i.e., $\Delta c_1 = 0$ and thus $\Delta c_2 = \overline{\Delta c}$.

Let $\overline{s}$ be the corresponding resource when

$$\Delta c_2 = \overline{\Delta c} = \frac{c_2 - c_1}{2}. \tag{65}$$

In this case, by (19), (20) and (65), we can derive that

$$\overline{s} = \frac{2(c-1)V}{(3+c)^2}.$$

Next, we seek to find out what the optimal subsidy scheme is when $s > \overline{s}$. When $\overline{\Delta c} > \frac{c_2 - c_1}{2}$, it is optimal to set $K(\Delta c_1) = 0$, which yields the optimal Δc_1 and Δc_2:

$$\Delta c_1 = \frac{1}{4}[2\overline{\Delta c} - (c_2 - c_1)] > 0; \tag{66}$$
$$\Delta c_2 = \frac{1}{4}[2\overline{\Delta c} + (c_2 - c_1)] > 0. \tag{67}$$

[40] Note the entire budget will be used up as for example holding Δc_1 constant, a small increase in TE can be obtained by a small increase in $\overline{\Delta c}$ and that (63) implies that this leads to a small increase in total resource; thus the initial solution could not have been optimal if the budget was not exhausted.

The following two conditions must be satisfied:

$$c_1' = c_1 - \Delta c_1 = \frac{1}{4}[3c_1 + c_2 - 2\overline{\Delta c}] \geq 0; \tag{68}$$

$$c_2' = c_2 - \Delta c_2 = \frac{1}{4}[3c_2 + c_1 - 2\overline{\Delta c}] \geq 0. \tag{69}$$

When

$$\frac{c_2 - c_1}{2} < \overline{\Delta c} \leq \frac{3c_1 + c_2}{2},$$

both (68) and (69) hold, so (66) and (67) are the optimal solutions for maximizing total effort. When $\overline{\Delta c} = \frac{3c_1+c_2}{2}$, by (24),

$$\overline{TE} = \frac{2V}{c_2 - c_1},$$

and by (66) and (67), $\Delta c_1 = c_1$ and $\Delta c_2 = \frac{1}{2}(c_1 + c_2)$; therefore, using (61) we have

$$s = \frac{2c_1 V}{(c_2 - c_1)} = \frac{2V}{(c-1)}.$$

It follows that when $\overline{s} < s < \overline{s}^*$ where

$$\overline{s}^* = \frac{2c_1 V}{(c_2 - c_1)} = \frac{2V}{(c-1)},$$

the two players are subsidized simultaneously.[41]

Also notice that when $s = \overline{s}^*$, $c_1' = c_1 - \Delta c_1 = 0$, which means the strong player will be fully subsidized in equilibrium.

When $s > \overline{s}^*$, $\overline{\Delta c} > \frac{3c_1+c_2}{2}$, and $K(\Delta c_1)$ must be negative as

$$\begin{aligned} K(\Delta c_1) &< \left[c_2 - c_1 - 2\left(\frac{3c_1 + c_2}{2}\right) + 4\Delta c_1\right] \\ &= 4(\Delta c_1 - c_1) \leq 0. \end{aligned}$$

In this case in order to minimize (63), it is optimal to set Δc_1 to its maximal value, i.e., $\Delta c_1 = c_1$ and thus $\Delta c_2 = \overline{\Delta c} - c_1$. When $\Delta c_1 = c_1$, however, there are multiple equilibria where the strong player could exert an arbitrarily large effort which makes the total amount of subsidies exceed s. To avoid this problem, we can simply assume that player 1 puts in the minimum effort necessary to "shut out" player 2. Alternatively, the contest designer can approximate the optimum by setting $\Delta c_1 = c_1 - \varepsilon$ where ε (> 0) is arbitrarily small, so the strong player is almost fully subsidized and the weak player exerts arbitrarily little effort.[42]

[41] Note that by (66) and (67) we have that when $\overline{s} < s \leq \overline{s}^*$, $c_2' - c_1' = (c_2 - c_1)/2$, so that the difference of the two players' costs is halved. In fact when s increases from 0 to $\overline{s}$, $(c_2' - c_1')$ decreases from $(c_2 - c_1)$ to $(c_2 - c_1)/2$; for $\overline{s} < s < \overline{s}^*$, $(c_2' - c_1')$ remains unchanged at $(c_2 - c_1)/2$; when s exceeds $\overline{s}^*$, $(c_2' - c_1')$ is decreasing in s, but only approaches zero as $s \to \infty$.

[42] One might think that when s attains some threshold level, Δc_2 will reach c_2, i.e., the two players are both fully subsidized. However, this cannot happen: (24) implies that TE would go to infinity when Δc_2 approaches c_2 so the required subsidy would also go to infinity.

B1. We have shown that if adding to the prize is not allowed, when $s \leq \overline{s}$, it is optimal to use all the resource s to subsidize the weak player. Now, we seek to show that when adding to the prize is allowed, the optimal subsidy scheme from Part A (when adding to the prize is not allowed) still maximizes total effort. We compare the effect of only subsidizing the weak player to that of subsidizing the weak player and adding to the prize simultaneously. Suppose resource s_2 is used to subsidize the weak player and $s_P = s - s_2$ is used to add to the prize (with $s_1 = 0$). Hence

$$s_2 = \Delta c_2 e_2' = \Delta c_2 \frac{c_1(V + s - s_2)}{(c_1 + c_2 - \Delta c_2)^2}. \tag{70}$$

Solving (70) for Δc_2 yields

$$\Delta c_2 = \frac{c_1(s + s_2 + V) + 2c_2 s_2 - \sqrt{c_1}\sqrt{V + s - s_2}\sqrt{c_1(s + 3s_2 + V) + 4c_2 s_2}}{2s_2}. \tag{71}$$

Substituting (71) into (24):

$$TE = \frac{2s_2(s - s_2 + V)}{\sqrt{c_1}\sqrt{s - s_2 + V} + \sqrt{4c_2 s_2 + c_1(s + 3s_2 + V)} - c_1(s - s_2 + V)}.$$

Differentiating with respect to s_2 and setting this equal to zero yields the following solution:

$$s_2^* = \frac{\left(1 + 2c - \sqrt{1 + c}\right)(s + V)}{3 + 4c} > 0. \tag{72}$$

Recall that TE is initially increasing as s_2 increases from zero, so $s_2 = s_2^*$ is the point at which TE is maximized when $s_2 \leq s_2^*$ (as it is the first turning point).[43] Thus, it can be concluded that when $s \leq s_2^*$, i.e.,

$$s \leq \left(1 - \frac{1}{\sqrt{1 + c}}\right) V, \tag{73}$$

we should spend all resource s on subsidizing the weak player rather than increasing the prize.

For any $c > 1$, it can be verified that

$$\overline{s} = \frac{2(c - 1)V}{(3 + c)^2} < \left(1 - \frac{1}{\sqrt{1 + c}}\right) V,$$

so when $s \leq \overline{s}$, adding to the prize is dominated by subsidizing the weak player. Therefore, when adding to the prize is allowed, the optimal subsidy scheme is still subsidizing the weak player provided $s \leq \overline{s}$.

B2**.** When $\overline{s} < s < \overline{s}^*$, recall that the optimal subsidy scheme (ignoring the possibility of increasing the prize) is to subsidize both players simultaneously. For s in this range, suppose

[43] So for $s < s_2^*$ it is optimal to spend all s subsidizing the weak player.

that the following scheme maximizes total effort: adding resource $s_P \geq 0$ to the prize, so the new prize is $V^n := V + s_P$, and the total resource for subsidies is $s^n := s - s_P$. The corresponding version of (3) and (4) then is:

$$s_i = (c_i - c_i')e_i', \text{ where } e_i' = \frac{c_j' V^n}{(c_1' + c_2')^2},$$

$i, j = 1, 2, i \neq j$. Any choice of (c_1', c_2') implies unique values for s_1, s_2, s_P and effort levels. It is more convenient to work directly with (c_1', c_2'):

$$e_i' = \frac{c_j' (V + s)}{c_1' (c_1' + c_2) + c_2' (c_2' + c_1)}, \tag{74}$$
$$s_i = \frac{c_j' (c_i - c_i') (V + s)}{c_1' (c_1' + c_2) + c_2' (c_2' + c_1)},$$

$i, j = 1, 2,\ i \neq j$ (after manipulation). We consider maximizing total effort $e_1' + e_2'$ subject to the constraint that $s_1 + s_2 = s^n$. Using (74), define the Lagrangian function

$$L (c_1', c_2', \mu) := e_1' + e_2' + \mu(s^n - s_1 - s_2).$$

First order conditions yield:

$$c_i' = \frac{2c_i(s^n + V^n) - 2c_j s^n + \sqrt{2}\phi}{4(2s^n + V^n)},\ i = 1, 2,\ j \neq i, \tag{75}$$

$$\mu = \frac{\sqrt{2}[(c_1 + c_2)^2 V^n - (c_2 - c_1)^2 s^n] - 2(c_1 + c_2)\phi}{(c_2 - c_1)^2 \phi}, \tag{76}$$

where

$$\phi := \sqrt{2c_1 c_2 (V^n)^2 - (c_2 - c_1)^2 s^n V^n}.$$

Note that $s < \overline{s}^*$ implies $2c_1c_2(V^n)^2 - (c_2 - c_1)^2 s^n V^n > 0$, since this would be most likely to fail if $s^n = s = \overline{s}^*$, at which point it follows that $V^n = V$ and $\phi = \sqrt{2c_1c_2V^2 - (c_2 - c_1)^2 sV} = \sqrt{2}c_1 V > 0$. Therefore, when $s^n < s \leq \overline{s}^*$, s^n will be smaller than $\overline{s}^*$ and V^n will be bigger than V, so we always have $\phi > 0$.

Suppose that the optimum occurs with $s_P > 0$. Recall that $\overline{s}^* = 2V/(c - 1)$; here the prize is increased to $V + s_P$, so in this case $s < \overline{s}^*$ implies that $s < 2\,(V + s_P)\,/(c - 1)$ and consequently both $c_i' > 0$ and $s_i > 0$, thus nonnegativity constraints are not binding. Then setting $s^n = s - s_P$ in the above problem would lead to the optimum outcome and (75) and (76) must hold. However it is tedious but straightforward to check that for $s < \overline{s}^*$, the numerator in the right hand side of (76) is positive, so $\mu > 0$. This implies that an increase in s^n, i.e., a reduction in s_P, would increase effort, contrary to the assumption of optimality.

B3. Finally, consider the case where $s \geq \overline{s}^*$. In the equilibrium under the optimal subsidy scheme, the strong player will always be almost fully subsidized and the weak player makes zero effort in equilibrium. Thus, $s = c_1 e_1' = c_1 TE'$, so $TE' = s/c_1$. We will show that the optimal subsidy scheme still maximizes total effort when adding to the prize is allowed. Assume again that to maximize total effort, some resource $s_P > 0$ should be added to the prize; we establish this will lead to a contradiction, i.e., the optimal $s_P = 0$.

With the optimal $s_P > 0$ fixed, when

$$s^n \geq \overline{s}^{n*} := \frac{2V^n}{(c-1)},\ TE' = \frac{s^n}{c_1} = \frac{s - s_P}{c_1}.$$

Clearly total effort is increased if s_P is reduced, contrary to the assumed optimality. Likewise, when $s^n < \overline{s}^{n*}$, optimal subsidies imply $c_i' > 0$, $i = 1, 2$, and the previous argument applies to show a reduction in s_P increases total effort.

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