

Self-Fulfilling Price Cycles

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Abstract

This paper presents a model of a self-fulfilling price cycle in an asset market. Price oscillates deterministically even though the underlying environment is stationary. The mechanism that we uncover is driven by endogenous variation in the investment horizons of the different market participants, informed and uninformed.

On even days, the price is high; on odd days it is low.

On even days, informed traders are willing to jettison their good assets, knowing that they can buy them back the next day, when the price is low. The anticipated drop in price more than offsets any potential loss in dividend. Because of these asset sales, the informed build up their cash holdings. Understanding that the market is flooded with good assets, the uninformed traders are willing to pay a high price. But their investment horizon is longer than that of the informed traders: their intention is to hold the assets they purchase, not to resell.

On odd days, the price is low because the uninformed recognise that the informed are using their cash holdings to cherry-pick good assets from the market. Now the uninformed, like the informed, are investing short-term. Rather than buy-and-hold as they do with assets purchased on even days, on odd days the uninformed are buying to sell.

Notice that, at the root of the model, there lies a credit constraint. Although the informed are flush with cash on odd days, they are not deep pockets. On each cherry that they pick out of the market, they earn a high return: buying cheap, selling dear. However they don't have enough cash to strip the market of cherries and thereby bid the price up.

1 Introduction

Can the apparent excess volatility of asset prices be explained from within the model – that is to say, endogenously, without exogenous variation? It is known that bubbles can be part of a *stochastic* equilibrium. In this paper we set ourselves the challenge of generating *deterministic* price movement in a perfectly stationary environment.

We show that it is possible. In our model the dynamic interaction of adverse selection (we have informed and uninformed traders) and credit constraints (the informed traders do not have deep pockets) can endogenously generate variation in the investment horizons of the different market participants. With these simple ingredients, we demonstrate a saw-tooth equilibrium in which prices deterministically rise one day and fall the next, *ad infinitum*. On even days, the price is high; on odd days it is low.

On even days, informed traders are willing to jettison their good assets, knowing that they can buy them back the next day, when the price is low. The anticipated drop in price more than offsets any potential loss in dividend. Because of these asset sales, the informed build up their cash holdings. Understanding that the market is flooded with good assets, the uninformed traders are willing to pay a high price. But their investment horizon is longer than that of the informed traders: their intention is to hold the assets they purchase, not to resell.

On odd days, the price is low because the uninformed recognise that the informed are using their cash holdings to cherry-pick good assets from the market. Now the uninformed, like the informed, are investing short-term. Rather than buy-and-hold as they do with assets purchased on even days, on odd days the uninformed are buying to sell.

Notice that, at the root of the model, there lies a credit constraint. Although the informed are flush with cash on odd days, they are not deep pockets. On each cherry that they pick out of the market, they earn a high return: buying cheap, selling dear. However they don't have enough cash to strip the market of cherries and thereby bid the price up.

The literature on adverse selection, starting with Akerlof [1970], is vast, and it would be unwise to review it here. Of particular relevance are the papers by Hendel and Lizzeri [1999] and Hendel et al. [2005] on dynamic adverse selection in the context of durable goods markets.

Eisfeldt [2004] and Guerrieri and Shimer [2013] use adverse selection to model liquidity in a dynamic infinite horizon setting. Aggregate dynamics within their models result from aggregate shocks. A key difference, then, between these papers and our own is that the aggregate dynamics in our model arise completely endogenously, in a perfectly stationary environment.

In Section II, we introduce the formal model. In Section III, we examine a constant

price equilibrium. In section IV, we construct a self-fulfilling price cycle equilibrium. In section V, we provide a numerical example.

2 Model

The economy is discrete time, with a durable generic consumption good, fruit, and a single kind of asset, trees. At the start of each day trees mature with probability μ . The maturation process is *i.i.d.* across trees, age and time. When trees mature they bear fruit and immediately die. Trees that do not mature do not bear fruit and do not die.

There are two types of tree: high quality “cherry” trees and low quality “lemon” trees. A fraction γ of trees are cherry. When trees mature, lemon trees bear $l > 0$ (generic) fruit and cherry trees bear $h = l + \Delta$ (generic) fruit, where $\Delta > 0$.

There is a continuum of risk-neutral traders. At the end of each day traders die with probability δ . Agents learn earlier in the day, after trees have matured, if they are to die at the end of the day. Death is *i.i.d.* across traders, age and time. Agents consume fruit, only once, just before they die.

There are two types of trader: “informed” traders and “uninformed” traders. Informed traders can determine whether an unmaturing tree is a cherry or a lemon; uninformed traders cannot. A fraction θ of traders are informed.

Each day – after the maturation of a fraction μ of the existing trees, but before a fraction δ of traders learn if they will die at the end of the day – a mass α of new traders are born endowed with a mass τ of new, unmaturing, trees. A fraction γ of these new trees are cherry and a fraction θ of these new traders are informed. The overnight steady-state stock T of trees (of which γT are cherry) solves $\mu T = \tau$. And the overnight steady-state mass A of traders (of whom θA are informed) solves $\delta A = \alpha$.

There is a daily competitive spot market in which trees are traded for fruit. The market occurs after traders learn if they will die at the end of the day. That is, the timing on each day is as follows:

1. A fraction μ of the existing trees mature: they yield fruit and die.
2. A mass α of new traders (of whom a fraction θ are informed) are born endowed with a mass τ of new, unmaturing, trees (of which a fraction γ are cherry).
3. A fraction δ of traders – including a fraction δ of the newly-born¹ – learn that they are dying today (see 5 below).
4. A market occurs in which trees are traded for fruit.

¹These traders live short lives.

5. The dying traders consume their fruit and die.

We can save on notation by assuming $\tau = \mu$ and $\alpha = \delta$, so that $T = 1$ and $A = 1$.

This model typically has several equilibria. Given the stationary nature of the environment, it is unsurprising that there is an equilibrium where price is constant through time; see section 2 below.

We do not try to characterise all the non-constant-price equilibria. However, of particular interest to us is an equilibrium where the price deterministically oscillates: a high price on (e.g.) even days and a low price on odd days. We analyse this “saw-tooth” equilibrium in section 3. It is to this surprising equilibrium that we wish to draw the reader’s attention.

There are features that are common to all the equilibria we study – i.e., common to a constant-price equilibrium and a saw-tooth equilibrium. Crucially, we assume that informed traders are able to cherry pick in the market – they only purchase cherry trees, to the extent that their budgets allow. Uninformed traders purchase the residual supply of trees. Of the trees purchased by the uninformed on day t , let q_t be the equilibrium fraction that are cherry.

The market price p_t will reflect the uninformed buyers’ (rational) beliefs about q_t . That is, p_t is determined by an indifference condition for an uninformed buyer: between holding p_t fruit and purchasing a tree of “quality” q_t . In all the equilibria we examine in sections 3 and 4, we assume that the parameters are such that the informed buyers as a whole cannot afford to purchase all the cherry trees being supplied to the market, so $q_t > 0$.

Two behaviour patterns are obvious. First, a dying trader sells all his trees so as to maximise his fruit consumption before he dies. Second, on the day she is born, an informed trader sells all her endowment of lemon trees: if she uses the proceeds to purchase cherry trees then she is in effect using the market to replace lemon with cherry. (To simplify our diagrams, as an accounting convention we assume that informed newborns always *first* replace lemon with cherry, even if they have discovered they are going to die later that day and so will immediately resell these cherry trees.)

Finally, we make five assumptions on parameters the reasons for which will not be apparent to the reader until later in the paper:

Assumption 1:

$$0 < a_0 \equiv \theta\mu \{l(1 - \gamma)(1 - \delta) + l(1 - \delta)^2\mu + \Delta\gamma(1 - \delta)^2\mu\} - \gamma\delta[l\delta - \Delta(1 - \delta)\mu] \tag{1}$$

Assumption 2:

$$\left[1 - \frac{\delta}{(1-\delta)\mu}\right] - \theta > 0 \quad (2)$$

Assumption 3:

$$0 < \tilde{a}_0 \equiv \theta\mu(1-\delta)^2 \left\{ (2-\delta)l + (1-\delta)\mu\Delta + \gamma[1 + (1-\delta)(1-\mu)] \left[1 - \frac{\delta(1-\mu)}{2-\mu}\right] \Delta \right\} \\ - [\gamma\delta - (1-\delta)(1-\gamma)\theta\mu] \left\{ \delta(2-\delta) \left[l + \gamma(1-\mu) \left(1 - \frac{\delta(1-\mu)}{2-\mu}\right) \Delta \right] - (1-\delta)^2\mu\Delta \right\} \quad (3)$$

Assumption 4:

$$\gamma > \frac{1}{1 + (1-\delta)(1-\mu)} \quad (4)$$

Assumption 5:

$$0 < \theta\mu(1-\delta)^2 \left\{ (2-\mu)(2-\delta)(l+\Delta) - \frac{2(1-\gamma)\Delta[1 + (1-\delta)(1-\mu)]^2}{(2-\mu)(1-\delta)} \right\} \\ - \left\{ \delta^2(2-\delta)(l+\Delta) - \frac{\delta(1-\gamma)\Delta[1 + (1-\delta)(1-\mu)][1 - (1-\mu)^2(1-\delta)^2]}{1-\delta)(1-\mu)(2-\mu)} \right\} \quad (5)$$

An analytical proof that parameters consistent with these assumptions exist is left to the appendix.

3 Constant Price Equilibrium

In this equilibrium the price and market quality of trees is constant: $p_t = p^*$ and $q_t = q^*$ for all t . Dying traders sell all their trees; new born informed traders switch any lemon trees they carry for cherry (as described in section 2 above) – no other trees are sold. Informed traders exchange all their fruit for cherry trees sold by the dying traders. Uninformed traders purchase the residual stock of trees sold by the dying and the lemons sold by the new-born informed traders. q^* is the proportion of these trees bought by the uninformed that are cherry.

3.1 Incentive Compatibility and Price

The informed and uninformed equilibrium behaviour follows from the fact that p^* is the value of a market tree to an uninformed trader. The supply of market trees – the

residual trees sold after cherry picking by the informed – is positive. Therefore, from the indifference of the uninformed between market trees and p^* fruit, the net purchase of market quality trees by the uninformed is strictly positive i.e., the uninformed do not sell their market quality trees.

Neither do the uninformed sell trees with which they are born as these trees are strictly preferred to p^* fruit. To see this we first derive p^* from its equivalence with an uninformed trader's valuation of a market quality tree:

$$\begin{aligned} p^* &= \mu(l + q^*\Delta) + (1 - \mu)p^* \\ &= l + q^*\Delta. \end{aligned} \tag{6}$$

Where $(l + q^*\Delta)$ is the expected yield if the tree matures and p^* is the value of the tree if it does not mature.

A proportion γ of trees with which traders are born are cherry – a strictly greater proportion of cherry than for market trees. If a trader holds a tree with which they are born it can be sold for a price p^* if it does not mature. The value of such a tree is then bounded below by

$$\mu(l + \gamma\Delta) + (1 - \mu)p^*,$$

which is strictly greater than p^* as defined in equation (6). Therefore uninformed traders strictly prefer the trees with which they are born to p^* fruit. The above argument holds *a fortiori* for cherry trees, i.e. informed traders strictly prefer cherry trees to p^* fruit. Therefore informed traders exchange all their fruit for cherry trees and never sell cherry trees if they are not dying that day.

3.2 Market Clearing

As the informed traders hold only cherry trees overnight – call this stock N – the uninformed must hold the entire overnight stock of fruit in the economy – call this stock Y . At the equilibrium steady state of Y , Y^* , the total fruit consumed per day equals total fruit produced per day.

The equilibrium steady state of N , N^* , is where the outflow of cherry trees from informed traders' overnight stock is equal to the inflow. Of the $(1 - \delta)N^*$ trees held by the non-dying, a fraction μ are lost to maturation. δ of the N^* cherry trees are sold each period by the dying. The maturing cherry trees held by the non-dying informed traders yield $(1 - \delta)\mu N^*(l + \Delta)$ fruit – this fruit is then exchanged for cherry trees. The non-dying new-born informed traders replace their lemons with cherry adding $(1 - \delta)\theta\mu$ cherry trees to the stock of informed trees each day. The steady state condition for N^* is then:

$$\begin{aligned}
(1 - \delta)\mu N^* &+ \delta N^* &= &(1 - \delta)\mu(l + \Delta)\frac{N^*}{p^*} + (1 - \delta)\theta\mu \\
\text{Maturing trees} &\text{Trees held} &\text{Trees bought} &\text{Trees of} \\
\text{held by the non-} &\text{by the dying} &\text{from proceeds of} &\text{non-dying} \\
\text{dying informed} &\text{informed.} &\text{maturing trees.} &\text{new-borns.}
\end{aligned} \tag{7}$$

We can define all equilibrium stocks and flows in the economy in terms of N^* and Y^* . These stocks and flows are illustrated in figure 1 below:

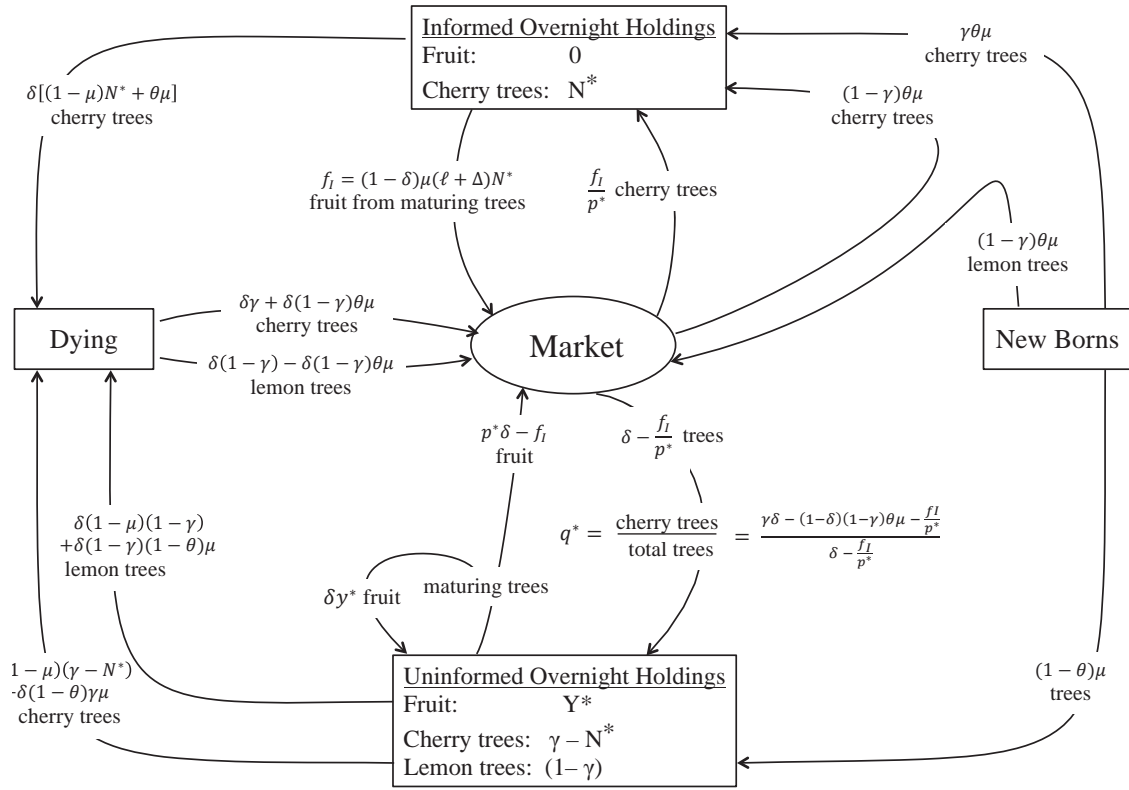


Figure 1: Stocks and Flows in the Constant Price Equilibrium

Markets must clear. In figure 1 we see the total number of trees sold by the dying and the newborn is:

$$\begin{aligned}
\text{Tree Sales} &= \delta &+ &(1 - \gamma)(1 - \delta)\theta\mu; \\
&\text{From dying} &&\text{Lemon trees from} \\
&\text{informed and} &&\text{non-dying} \\
&\text{uninformed.} &&\text{informed new-borns.}
\end{aligned} \tag{8}$$

and the quantity of these trees that are cherry is:

$$\begin{aligned} \text{Cherry Sales} &= \gamma\delta. \\ &\text{Cherry trees from dying} \\ &\text{informed and uninformed.} \end{aligned} \tag{9}$$

The informed buy only cherry trees:

$$\begin{aligned} \text{Informed Cherry Purchases} &= (1 - \delta)\mu(l + \Delta)\frac{N^*}{p^*} + (1 - \gamma)(1 - \delta)\theta\mu \\ &\text{Cherry trees bought from} && \text{Cherry trees bought} \\ &\text{proceeds of maturing} && \text{by non-dying} \\ &\text{cherry trees held by non-} && \text{informed} \\ &\text{dying informed.} && \text{new-borns.} \end{aligned} \tag{10}$$

The uninformed purchase the residual trees, a fraction of which are cherry:

$$\begin{aligned} \text{Uninformed Cherry} &= q^* \left\{ \delta + (1 - \gamma)(1 - \delta)\theta\mu - \left[(1 - \delta)\mu(l + \Delta)\frac{N^*}{p^*} + (1 - \gamma)(1 - \delta)\theta\mu \right] \right\} \\ \text{Purchases} &\qquad \qquad \qquad \text{Trees sold} \qquad - \qquad \text{Informed cherry purchases} \\ &= q^* \left[\delta - (1 - \delta)\mu(l + \Delta)\frac{N^*}{p^*} \right]. \end{aligned} \tag{11}$$

Hence, the market clearing condition for cherry trees² is:

$$\gamma\delta = (1 - \delta)\mu(l + \Delta)\frac{N^*}{p^*} + (1 - \gamma)(1 - \delta)\theta\mu + q^* \left[\delta - (1 - \delta)\mu(l + \Delta)\frac{N^*}{p^*} \right]$$

$$\begin{aligned} \text{Cherry sales.} &\qquad \qquad \text{Informed cherry purchases.} && \text{Uninformed cherry purchases.} \\ &&& \tag{12} \end{aligned}$$

Collecting the $\frac{N^*}{p^*}$ terms in the market clearing condition yields:

$$(\gamma - q^*)\delta - (1 - \gamma)(1 - \delta)\theta\mu = (1 - q^*)(1 - \delta)\mu(l + \Delta)\frac{N^*}{p^*} \tag{13}$$

²Which also implies market clearing for lemon trees. Total lemon tree sales is: Lemon Sales = Tree Sales – Cherry Sales. Total lemon tree purchases is: Lemon Purchases = (1 – q*)[Tree Sales – Informed Cherry Purchases]. Letting Lemon Sales = Lemon Purchases and rearranging yields equation (12).

and doing the same for the steady state stock of informed cherry holdings given by equation (7) yields

$$\left\{ \frac{l + q^* \Delta}{l + \Delta} [\delta + (1 - \delta)\mu] - (1 - \delta)\mu \right\} (l + \Delta) \frac{N^*}{p^*} = (1 - \delta)\theta\mu \quad (14)$$

where $l + q^* \Delta$ comes from equation (6) for p^* . Using (13) to substitute out $\frac{N^*}{p^*}$ in (14) and rearranging gives the quadratic:

$$\begin{aligned} \Phi(q^*) &\equiv (1 - \delta)^2 \mu^2 \theta (1 - q^*) \\ &\quad - \left\{ \frac{l + q^* \Delta}{l + \Delta} [\delta + (1 - \delta)] - (1 - \delta)\mu \right\} [\delta(\gamma - q^*) - (1 - \gamma)(1 - \delta)\theta\mu] \\ &= 0 \end{aligned} \quad (15)$$

Given the equilibrium behaviour of informed and uninformed traders we require that the solution to this quadratic lies in the interval of $[0, \gamma]$.

The quadratic function $\Phi(\cdot)$ can be written more compactly as:

$$\Phi(q) = a_2 q^2 + a_1 q + a_0; \quad (16)$$

where

$$a_2 = \delta \frac{\Delta}{l + \Delta} [\delta + (1 - \delta)\mu] > 0. \quad (17)$$

The coefficient on the square term, a_2 , is positive and a_0 is less than zero by assumption (Assumption 1 section 2). Therefore, if $\Phi(\gamma) > 0$, there exists a unique $q^* \in [0, \gamma]$ such that $\Phi(q^*) = 0$. Define q_N such that:³

$$\delta(\gamma - q_N) - (1 - \gamma)(1 - \delta)\theta\mu = 0. \quad (18)$$

From $a_0 < 0$ it follows that

$$\begin{aligned} -l\gamma\delta^2 + \theta\mu l(1 - \gamma)(1 - \delta) &> 0 \\ \Rightarrow \delta\gamma - (1 - \gamma)(1 - \delta)\theta\mu &> 0 \end{aligned} \quad (19)$$

³ q_N because from (13) $N^* = 0$ where $q = q_N$.

and therefore

$$\gamma > q_N > 0. \quad (20)$$

Evaluate the sign of $\Phi(\cdot)$ at 0 and q_N :

$$\Phi(0) = a_0 < 0 \quad (21)$$

$$\Phi(q_N) = (1 - \delta)^2 \mu^2 \theta (1 - q_N) > 0. \quad (22)$$

Therefore there exists a unique $q^* \in [0, \gamma]$ such that $\Phi(q^*) = 0$ as shown in figure 2 below:

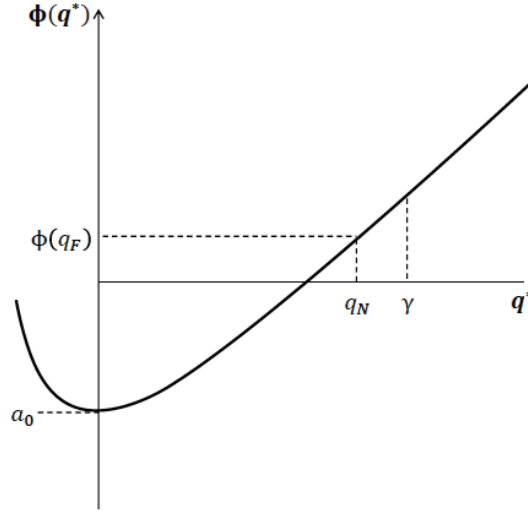


Figure 2

Next, it is shown that $N^* > 0$. Examining the right hand side of the quadratic in (15), it can be seen that when $q^* = 0$ the first term is positive and so too is the final term in brackets. As $\Phi(0) < 0$ it must be the case that

$$\frac{l + q^* \Delta}{l + \Delta} [\delta + (1 - \delta)] - (1 - \delta) \mu > 0. \quad (23)$$

Which from equation (14) implies that

$$\frac{N^*}{p^*} > 0$$

and therefore $N^* > 0$ for $q^* \in [0, \gamma]$.

The stock of uninformed cherry trees held over night, $(\gamma - N^*)$, must also be greater than zero. In steady state the uninformed lose as many cherry trees as they gain. Hence, $(\gamma - N^*)$ solves

$$\begin{aligned}
(\gamma - N^*)[\delta + (1 - \delta)\mu] &= \gamma(1 - \delta)(1 - \theta)\mu & + & q^* \left[\delta - (1 - \delta)\mu(l + \Delta)\frac{N^*}{p^*} \right] \\
\text{Cherry lost to death} & \quad \text{Cherry held by} & & \text{Uninformed cherry purchases.} \\
\text{and maturation.} & \quad \text{undying newborns.} & & \\
\end{aligned} \tag{24}$$

If the right hand side is positive then $\gamma - N^*$ is positive. The number of cherry trees gained by the uninformed each period is positive if the number of cherry trees purchased by the uninformed is positive. Rearranging the market clearing condition (12) gives:

$$(\gamma - q^*) \left[\delta - (1 - \delta)\mu(l + \Delta)\frac{N^*}{p^*} \right] = (1 - \delta)(1 - \gamma) \left[\theta\mu + (l + \Delta)\frac{N^*}{p^*} \right] > 0, \tag{25}$$

which implies that uninformed cherry purchases are positive, i.e., their steady state stock of cherry trees is positive.

Finally, we show that the steady state stock of fruit held overnight by the uninformed is positive. The uninformed hold all the fruit in the economy. The steady state stock of fruit in the economy is where total daily fruit consumption equals total daily fruit production.

Dying traders consume their fruit held from the previous day, δY^* ; the fruit from selling their trees, δp^* ; and the fruit from their maturing trees held from the previous day, $\delta\mu(l + \gamma\Delta)$.⁴ Fruit production is due only to maturing trees and is equal to $\mu(l + \gamma\Delta)$. Thus Y^* solves

$$\delta Y^* + \delta p^* + \delta\mu(l + \gamma\Delta) = \mu(l + \gamma\Delta). \tag{26}$$

Fruit Consumed by the Dying Fruit from Maturing Trees

Substituting out p^* in (26) and rearranging implies that Y^* is positive if:

$$(1 - \delta)\mu(l + \gamma\Delta) - \delta(l + q^*\Delta) > 0.$$

$q^* \leq \gamma$ hence a sufficient condition for $Y^* > 0$ is:

$$(1 - \delta)\mu > \delta,$$

which follows immediately from assumption 2.

⁴Fruit is consumed only by the dying. Death is i.i.d. so the average dying trader holds the average of the economy.

4 Saw-Tooth Equilibrium

In this equilibrium the market price deterministically oscillates between low on odd days, p_1 , and high on even days, p_0 . Correspondingly, the quality of trees purchased from the market by the uninformed oscillates between low on odd days, q_1 , and high on even days, q_0 .

On even days all traders sell their trees. All trees sold on even days are purchased by non-dying uninformed traders – no tree is purchased by an informed trader. On even days all trees are sold, the average tree sold is the population average, and informed traders do not cherry pick the market. Hence, the quality of trees purchased by the uninformed on even days is the quality of the population, i.e., $q_0 = \gamma$.⁵

On odd days the dying traders sell all their trees and new born informed traders switch any lemon trees they carry for cherry (as described in section 2) these are the only trees sold on an odd day day. Death is i.i.d. so the trees sold by the dying are the population average. The non-dying informed traders exchange all their fruit for cherry trees – cherry picking trees sold by the dying. The uninformed purchase the residual stock of trees sold by the dying and the lemons sold by the new-born informed traders. Hence, the quality of trees purchased by the uninformed on odd days is lower than the population average, i.e., $q_1 < q_0 = \gamma$.

4.1 Incentive Compatibility and Prices

Surprisingly, on even days, non-dying informed traders sell trees they know to be cherry to non-dying uninformed traders who only believe the tree is cherry with probability γ . This trade is only possible because the differing investment horizons of the (non-dying) informed and uninformed traders.

On even days the non-dying informed traders have a shorter investment horizon than the non-dying uninformed: the return to informed traders from purchasing cherry trees is higher on odd days than on even days – so informed traders want to maximise the funds they have available for odd day investments. However, tree holdings suffer a temporary capital loss on odd days when prices drop from p_0 to p_1 – exactly when informed traders need funds for investment. Hence, informed traders are willing to pay a liquidity premium for fruit and take less than the value, $(l + \Delta)$, of a cherry tree's long-run dividend stream.

Non-dying uninformed traders, on the other hand, intend to hold even day trees for the long term – only selling the tree (in effect) when they die. The uninformed traders, if they do not die on the odd day, are unaffected by the temporary capital loss

⁵Note, that the behaviour of the non-dying uninformed is equivalent to holding their population average trees, those they hold from the previous even day or with which they were born, and only selling those trees purchased on odd days.

as they (mainly) value the asset for its dividend stream. The longer investment horizon of the uninformed then allows for gains from trade between the non-dying informed and uninformed traders despite the fact that the informed traders know they are selling their cherry trees for less than the ‘true’ value.

The volume of trade on odd days is lower than on even days as only the dying and new-born informed with lemons are selling. Trade between the dying and the non-dying occurs due to (the more conventional reason of) different preferences over trees and fruit. Trade between the non-dying informed and uninformed occurs because the informed are selling trees they know to be lemons to traders who believe that the trees are cherry with probability $q_1 > 0$.

Incentive compatibility of the uninformed strategy follows from the non-dying uninformed being indifferent between p_t fruit and a tree of quality q_t at date t . On even days they are indifferent between p_0 fruit and trees of quality γ and so, on even days, must strictly prefer p_0 fruit to trees of quality q_1 , so they sell odd day trees on even days. By the same argument, on odd days, they must strictly prefer trees of quality γ to p_1 fruit, so they don’t sell population trees on odd days.

Before showing incentive compatibility for the informed it is useful to have expressions for prices on odd and even days. The non-dying uninformed are making speculative purchases with an intent to resell the following day when prices are high. Their hope is that the tree won’t mature before they have a chance to sell it and claim the capital gains. Setting p_1 equal to the expected return then yields:

$$p_1 = \mu(l + q_1\Delta) + (1 - \mu)p_0 \tag{27}$$

The price on even days is lower than the value, $(l + \gamma\Delta)$, of a population tree’s dividend stream because the uninformed suffer a capital loss if they die on an odd day before the tree has matured. There are four disjoint outcomes for an uninformed trader holding a population tree:

Day of Event	Event	Expected Payoff	Probability
Odd	Tree Matures	$(l + \gamma\Delta)$	μ
Odd	Tree Doesn’t Mature Trader dies	p_1	$(1 - \mu)\delta$
Even	Tree Matures	$(l + \gamma\Delta)$	$(1 - \mu)(1 - \delta)\mu$
Even	Tree Doesn’t Mature	p_0	$(1 - \mu)^2(1 - \delta)$

Setting price equal to expected value yields

$$p_0 = \mu(l + \gamma\Delta) + (1 - \mu)\delta p_1 + (1 - \mu)(1 - \delta)\mu(l + \gamma) + (1 - \mu)^2(1 - \delta)p_0. \quad (28)$$

Substituting equation (27) for p_1 in (28) and rearranging gives the following expression for price on even days:

$$p_0 = l + \gamma\Delta - (\gamma - q_1)\frac{\delta(1 - \mu)}{2 - \mu}\Delta. \quad (29)$$

Which is the expected value of a population tree's dividend stream less the expected capital loss from having to sell in the event of dying on an odd day. It is worth noting for future reference that both p_0 and p_1 are affine functions of q_1 .

It is easy to check that the price does in fact oscillate. From equation (27), $p_0 > p_1$ if and only if $p_0 > (l + q_1\Delta)$. Subtracting $l + q_1\Delta$ from (29) and rearranging yields

$$p_0 - (l + q_1\Delta) = (\gamma - q_1)\left[1 - \frac{\delta(1 - \mu)}{2 - \mu}\right]\Delta,$$

which is greater than zero for $q_1 < \gamma$. In any feasible saw-tooth equilibrium the price on even days is higher than the price on odd days.

Returning to incentive compatibility, if an informed trader purchases a cherry tree on an odd day they can always sell on the subsequent day if it does not mature. Hence, the value of a cherry tree to an informed trader on an odd day is bounded below by

$$\mu(l + \Delta) + (1 - \mu)p_0 > p_1,$$

i.e. it is always optimal for non-dying informed traders, on odd days, to exchange all their fruit for cherry trees.

From this, on any odd day, informed traders either exchange their portfolio for cherry trees or, if they are dying, exchange their portfolio for fruit, which they then consume. In either event the informed traders utility is linearly increasing in the market value of their portfolio that day. Given, the optimal behaviour on odd days, the non-dying informed traders' decision problem on the even day is to maximise the expected value of their portfolio at price p_1 on the subsequent odd day.

Returns are linear so we can restrict attention to the decision of holding p_0 fruit versus holding a single cherry tree at the end of the even day. The expected market value of a cherry tree held over from the even day is $\mu(l + \Delta) + (1 - \mu)p_1$. Therefore, the informed traders' strategy is incentive compatible if and only if

$$p_0 \geq \mu(l + \Delta) + (1 - \mu)p_1.$$

Using the price equations, (27) and (29), to substitute out p_0 and p_1 above and rearranging gives the following incentive compatibility condition for informed traders:

$$0 \leq F \equiv -(1 - \gamma) - (\gamma - q_1)(1 - \delta)(1 - \mu). \quad (30)$$

We will show later that given the assumptions in section 2 this condition always holds.

4.2 Market Clearing

The total equilibrium stocks of fruit held over even nights by informed traders and uninformed traders are W_0 and Y_0 respectively. The total equilibrium stock of cherry trees held over odd nights by informed traders is $\frac{W_1}{p_1}$ and the total equilibrium stock of fruit held over odd nights is Y_1 .

Figure 3 shows the stocks and flows of trees and fruit on even days.

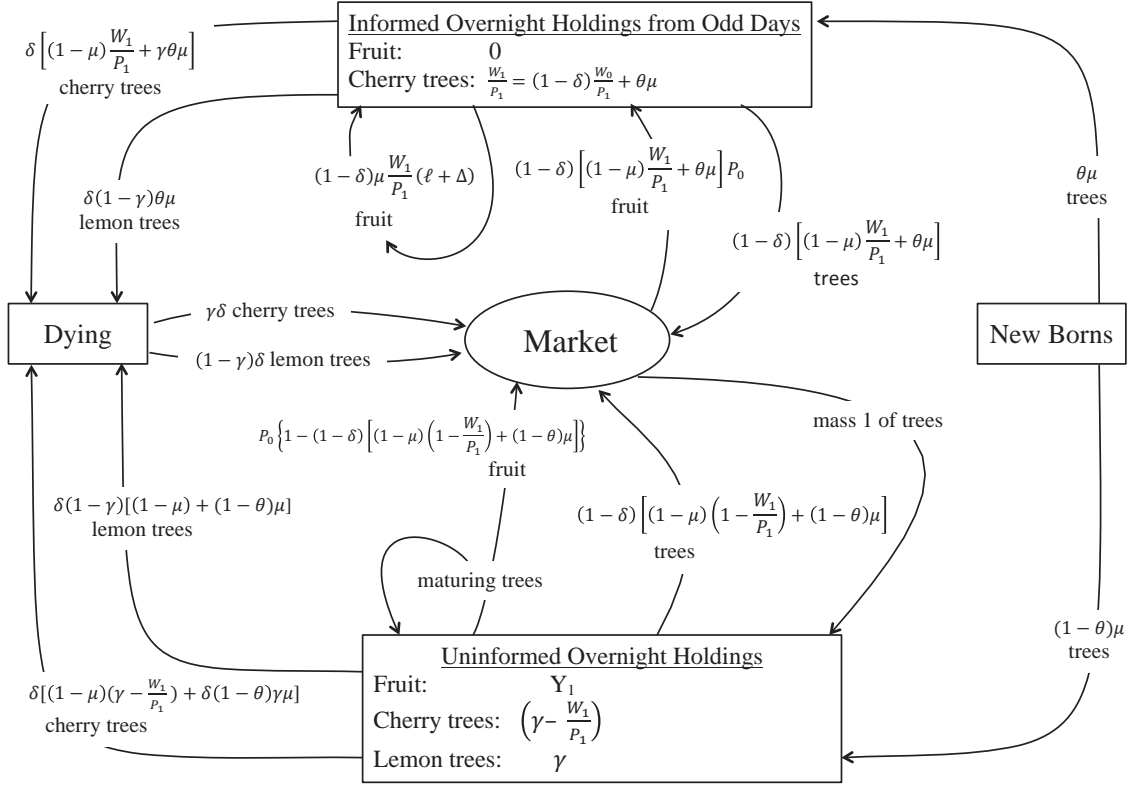


Figure 3: Stocks and Flows on Even Days

Looking at figure 3, the non-dying informed exchange all unmatured cherry trees – both those held over from the odd day and those carried by the new borns – for fruit.

The total fruit received is

$$(1 - \delta)\left[(1 - \mu)\frac{W_1}{p_1} + \theta\mu\right]p_0.$$

The non-dying informed also receive

$$(1 - \delta)\mu\frac{W_1}{p_1}(l + \Delta)$$

fruit from the cherry trees that matured on the morning of the even day. The informed had no fruit over the odd night and purchased no trees on the even day so the fruit holdings of the informed over even nights is

$$W_0 = (1 - \delta)\left\{\frac{W_1}{p_1}[\mu(l + \Delta) + (1 - \mu)p_0] + \theta\mu p_0\right\}. \quad (31)$$

Figure 4 shows the stocks and flows of trees and fruit on odd days.

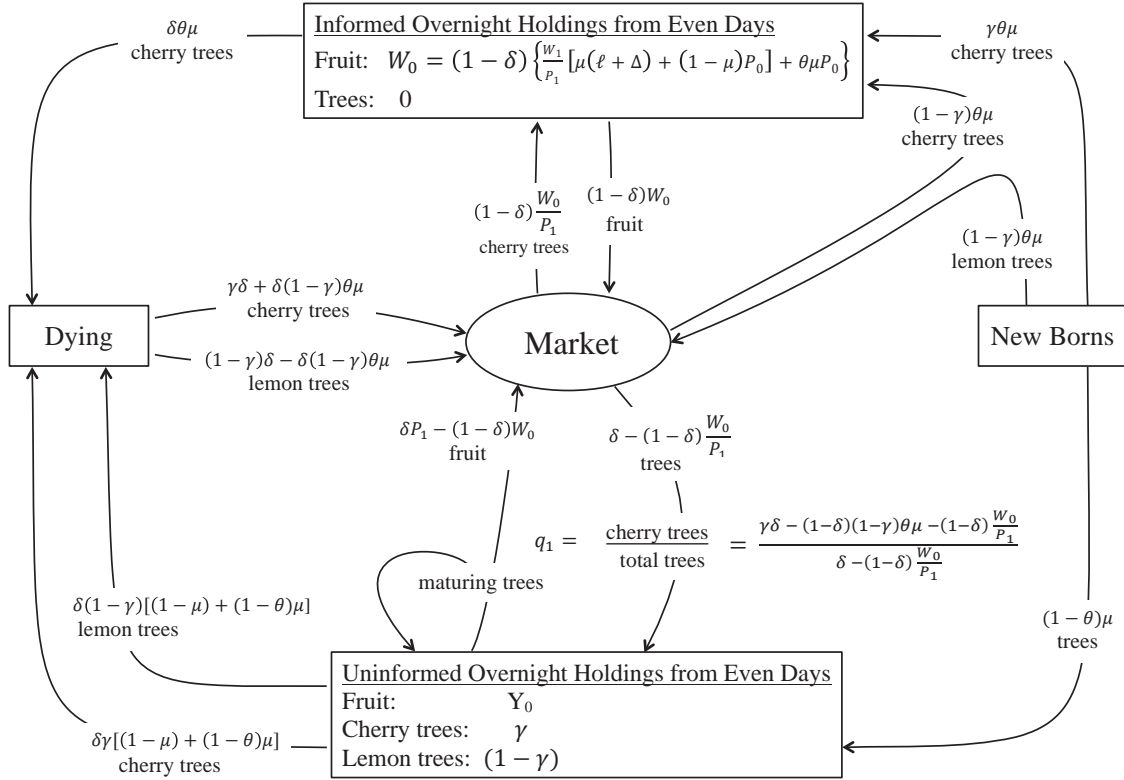


Figure 4: Stocks and Flows on Odd Days

Looking at figure 4, the non-dying exchange their fruit stock for

$$(1 - \delta)\frac{W_0}{p_1}$$

cherry trees on the odd day; and gain $(1-\delta)\theta\mu$ cherry trees from the non-dying newborn informed, of which a fraction $(1-\gamma)\theta\mu$ were acquired by using the market to replace lemon trees with cherry. The informed cherry tree holdings over odd nights then is

$$\frac{W_1}{p_1} = (1-\delta)\left(\frac{W_0}{p_1} + \theta\mu\right). \quad (32)$$

The informed hold no fruit over the odd night.

We can now go on to define the market clearing conditions on odd days. In figure 4 we see the total number of trees sold by the dying and the newborn is:

$$\begin{aligned} \text{Tree Sales} &= \delta && + && (1-\gamma)(1-\delta)\theta\mu; \\ &\text{from dying} && && \text{lemon trees from} \\ &\text{informed and} && && \text{non-dying} \\ &\text{uninformed} && && \text{informed new-borns} \end{aligned} \quad (33)$$

and the quantity of these trees that are cherry is:

$$\begin{aligned} \text{Cherry Sales} &= \gamma\delta. \\ &\text{cherry trees from dying} \\ &\text{informed and uninformed} \end{aligned} \quad (34)$$

The informed buy only cherry trees:

$$\begin{aligned} \text{Informed Cherry Purchases} &= (1-\delta)\frac{W_0}{p_1} && + && (1-\gamma)(1-\delta)\theta\mu \\ &\text{cherry trees bought from} && && \text{cherry trees bought} \\ &\text{fruit held over even} && && \text{by non-dying} \\ &\text{by non-dying} && && \text{informed} \\ &\text{informed} && && \text{new-borns} \end{aligned} \quad (35)$$

The uninformed purchase the residual trees, a fraction q_1 of which are cherry:

$$\begin{aligned} \text{Uninformed Cherry} &= q_1 \left\{ \delta + (1-\gamma)(1-\delta)\theta\mu - \left[(1-\delta)\frac{W_0}{p_1} + (1-\gamma)(1-\delta)\theta\mu \right] \right\} \\ \text{Purchases} & && \text{Trees Sold} && - && \text{Informed Cherry Purchases} \\ &= q_1 \left[\delta - (1-\delta)\frac{W_0}{p_1} \right]. \end{aligned} \quad (36)$$

Hence, the market clearing condition for cherry trees on odd days⁶ is:

$$\gamma\delta = (1 - \delta)\frac{W_0}{p_1} + (1 - \gamma)(1 - \delta)\theta\mu + q_1 \left[\delta - (1 - \delta)\frac{W_0}{p_1} \right].$$

Cherry Sales Informed Cherry Purchases Uninformed Cherry Purchases

(37)

Collecting the $\frac{W_0}{p_1}$ terms in the market clearing condition yields:

$$(\gamma - q_1)\delta - (1 - \gamma)(1 - \delta)\theta\mu = (1 - q_1)(1 - \delta)\frac{W_0}{p_1}. \quad (38)$$

Solving the steady state equations for W_0 and $\frac{W_1}{p_1}$, (31) and (32), in terms of $\frac{W_0}{p_1}$ yields:

$$\begin{aligned} & \frac{W_0}{p_1} \{p_1 - (1 - \delta)^2 [\mu(l + \Delta) + (1 - \mu)p_0]\} \\ & = \theta\mu(1 - \delta) \{(1 - \delta)\mu(l + \Delta) + [1 + (1 - \delta)(1 - \mu)]p_0\} \end{aligned} \quad (39)$$

Substituting out $\frac{W_0}{p_1}$ using (38), and rearranging yields:

$$\begin{aligned} \tilde{\Phi}(q_1) & \equiv (1 - \delta)^2(1 - q_1)\theta\mu \{(1 - \delta)\mu(l + \Delta) + [1 + (1 - \delta)(1 - \mu)]p_0\} \\ & - \{p_1 - (1 - \delta)^2 [\mu(l + \Delta) + (1 - \mu)p_0]\} [(\gamma - q_1)\delta - (1 - \delta)(1 - \gamma)\theta\mu] \\ & = 0 \end{aligned} \quad (40)$$

$\tilde{\Phi}(q_1)$ is quadratic in q_1 as both p_0 and p_1 are affine functions of q_1 :

$$\tilde{\Phi}(q_1) = \tilde{a}_2(q_1)^2 + \tilde{a}_1q_1 + \tilde{a}_0 \quad (41)$$

⁶Which also implies market clearing for lemon trees. Total lemon tree sales is: Lemon Sales = Tree Sales – Cherry Sales. Total lemon tree purchases is: Lemon Purchases = $(1 - q^*)$ [Tree Sales – Informed Cherry Purchases]. Letting Lemon Sales = Lemon Purchases and rearranging yields equation (37).

where

$$\begin{aligned}\tilde{a}_2 &= \frac{\delta\mu\Delta}{2-\mu} \left\{ (2-\mu) - [1 + (1-\delta)(1-\mu)](1-\delta)^2\theta(1-\mu) \right\} \\ &\quad + \delta^3\Delta(1-\mu)^2\frac{2-\delta}{2-\mu} \\ &> 0\end{aligned}\tag{42}$$

$\tilde{a}_2 > 0$ follows from $[1 + (1-\delta)(1-\mu)] < 2-\mu$ and by assumption 4 $\tilde{a}_0 < 0$. As in the constant price equilibrium, $\tilde{\Phi}(0) < 0$ and the co-efficient on the quadratic component is positive so we can show a unique $q_1 \in [0, \gamma]$ solves the quadratic if $\tilde{\Phi}(\gamma) > 0$.

Examining (40) it is clear that the argument used in the constant price equilibrium also applies here: $q_1 = q_N \in (0, \gamma)$ and $\tilde{\Phi}(q_N) > 0$ so there exists a unique $q_1 \in (0, \gamma)$ that solves the quadratic.

Examining (38) the left hand side is positive where $q_1 \in (0, q_N)$. Therefore where q_1 solves $\tilde{\Phi}(\cdot)$ the fruit holdings of the informed over even nights is positive, $W_0 > 0$. This also implies that the odd night cherry holdings for the informed is positive at the solution as it follows from (32) that:

$$(W_0 > 0) \Rightarrow \left(\frac{W_1}{p_1} > 0 \right).$$

Over even nights the uninformed hold the entire stock of trees in the economy. Over odd nights the uninformed hold the entire stock of lemon trees and all those cherry trees not held by informed. The stock of cherry trees held by the uninformed over odd nights is:

$$\gamma - \frac{W_1}{p_1} = (1-\delta)(1-\mu)\gamma + (1-\delta)\gamma(1-\theta)\mu + q_1 \left[\delta - (1-\delta)\frac{W_0}{p_1} \right]$$

Uninformed cherry trees held over odd nights	Residual stock from even night	Endowment of non-dying new-borns	Cherry trees purchased by uninformed
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(43)

The right hand side is positive if the uninformed purchase a positive amount of cherry trees. Rearranging the market clearing condition (37):

$$(\gamma - q_1) \left[\delta - (1-\delta)\frac{W_0}{p_1} \right] = (1-\gamma)(1-\delta) \left[\theta\mu + \frac{W_0}{p_1} \right].\tag{44}$$

The right hand side positive so it follows that

$$\delta - (1 - \delta) \frac{W_0}{p_1} > 0,$$

and hence cherry purchases of the uninformed on odd days are greater than zero. That is, the uninformed hold a positive stock of cherry trees over odd nights.

The uninformed hold weakly positive stocks of fruit over even and odd nights: $Y_0 > 0$ and $Y_1 > 0$. Over even nights

$$\begin{aligned} Y_0 = & (1 - \delta)Y_1 + \\ & + (1 - \delta)[\text{fruit from maturing trees of the uninformed}] \\ & - p_0[\text{Number of trees sold excluding the sales of the non-dying uninformed}] \end{aligned} \quad (45)$$

fruit. Before the market on even days the market value of all the trees in the economy not owned by the non-dying uninformed is the expression on the third line.

On the odd days, the non-dying uninformed keep fraction $(1 - \delta)$ of the fruit held over the even night. As they hold all trees in the economy over even nights the non-dying uninformed receive a fraction $(1 - \delta)$ of all the fruit from maturing trees in the economy. They also purchase all the trees sold by the dying less those bought by the informed. Over odd nights the uninformed fruit holding then is

$$Y_1 = (1 - \delta)Y_0 + (1 - \delta)\mu(l + \gamma\Delta) - p_1(\delta - (1 - \delta) \frac{W_0}{p_1}). \quad (46)$$

The uninformed hold over odd nights, as stated earlier, $[(1 - \delta)(1 - \mu) + (1 - \delta)(1 - \theta)\mu]$ trees of quality γ and

$$\delta - (1 - \delta) \frac{W_0}{p_1}$$

trees of quality q_1 . This implies that the uninformed maturing trees yield

$$\mu(1 - \delta)[(1 - \mu) + (1 - \theta)\mu](l + \gamma\Delta) + \mu[\delta - (1 - \delta) \frac{W_0}{p_1}](l + q_1\Delta)$$

fruit on even days. Excluding the sales of the non-dying uninformed, the trees sold on even days are those sold by the dying and those sold by the non-dying uninformed, which is a total of

$$\delta + (1 - \delta)^2(1 - \mu) \frac{W_0}{p_1} + (1 - \delta)^2(1 - \mu)\theta\mu + (1 - \delta)\theta\mu$$

trees.

Using these values and substituting for Y_1 in equation (45) using equation (46) we get

$$\begin{aligned}
Y_0 = & (1 - \delta) \left\{ (1 - \delta) Y_0 + (1 - \delta) \mu (l + \gamma \Delta) - p_1 \left[\delta - (1 - \delta) \frac{W_0}{p_1} \right] \right\} + \\
& + (1 - \delta) \left\{ \mu (1 - \delta) [(1 - \mu) + (1 - \theta) \mu] (l + \gamma \Delta) + \mu \left[\delta - (1 - \delta) \frac{W_0}{p_1} \right] (l + q_1 \Delta) \right\} \\
& - p_0 \left\{ \delta + (1 - \delta)^2 (1 - \mu) \frac{W_0}{p_1} + (1 - \delta)^2 (1 - \mu) \theta \mu + (1 - \delta) \theta \mu \right\}. \tag{47}
\end{aligned}$$

Substituting for p_1 above with equation (27) and rearranging implies $Y_0 \geq 0$ if and only if

$$\mu(1 - \delta)^2(l + \gamma \Delta)(2 - \theta \mu) \geq p_0[\delta + (1 - \delta)\theta \mu][1 + (1 - \delta)(1 - \mu)].$$

Given that $p_0 \leq (l + \gamma \Delta)$ a sufficient condition for $Y_0 \geq 0$ is

$$\mu(1 - \delta)^2(2 - \theta \mu) \geq [\delta + (1 - \delta)\theta \mu][1 + (1 - \delta)(1 - \mu)]$$

or,

$$\left[1 - \frac{\delta}{\mu(1 - \delta)} \right] - \theta > 0 \tag{48}$$

which is true by assumption 2.

Assumption 2 also implies that $Y_1 \geq 0$. The value of $Y_0 \geq 0$ that minimises Y_1 is $Y_0 = 0$. Putting this into equation (46) and rearranging yields

$$Y_1 = [(l + \gamma \Delta)(1 - \delta)\mu - \delta p_1] + (1 - \delta)W_0.$$

Therefore $Y_1 \geq 0$ if $(l + \gamma \Delta)(1 - \delta)\mu - \delta p_1 \geq 0$. Given $(l + \gamma \Delta) \geq p_1$ a sufficient condition for $Y_1 \geq 0$ is

$$\mu(1 - \delta) > \delta$$

, which is weaker than assumption 2.

It still remains to demonstrate that the equilibrium behaviour for the informed is incentive compatible. In particular that:

$$0 \leq F \equiv -(1 - \gamma) + (\gamma - q_1)(1 - \delta)(1 - \mu), \tag{49}$$

where $\tilde{\Phi}(q_1) = 0$ for some $q_1 \in (0, \gamma)$.

F is decreasing in q_1 . Define, q_F as that value of q_1 which implies $F = 0$:

$$q_F \equiv q_1 - \frac{(1 - \gamma)}{(1 - \delta)(1 - \mu)}. \quad (50)$$

Observe that $q_F > 0$ where

$$\gamma > \frac{1}{1 + (1 - \delta)(1 - \mu)}$$

which is assumption 4. From rearranging (18) and subtracting q_F :

$$q_N - q_F = \frac{(1 - \gamma)}{\delta(1 - \delta)(1 - \mu)} [\delta - (1 - \delta)^2 \mu(1 - \mu)\theta].$$

Which is greater than 0 as we know from assumption 1 that $\delta > (1 - \delta)^2 \mu$.

As $q_N > q_F$ we must demonstrate that $q_1 \in [0, q_F]$ solves $\tilde{\Phi}(q_1) = 0$. This can be done if it is demonstrated that $\tilde{\Phi}(q_F) > 0$, i.e.

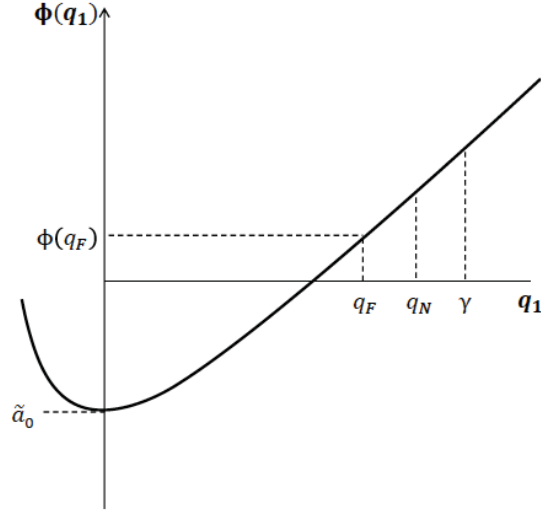


Figure 5

It is easy to see that assumption 5 implies $\tilde{\Phi}(q_F) > 0$ and hence there exists a unique incentive compatible $q_1 \in [0, \gamma]$ that solves $\tilde{\Phi}(q_1) = 0$.

5 A Numerical Example

Suppose an economy in which 1% of the population is informed; there are 50% more cherry trees than lemon and each cherry tree produces 50% more fruit than a lemon. Finally, one tree matures for every 10 that don't and 1 trader dies for every 20 that don't. Or:

$$\begin{aligned}\mu &= 1/11 \\ \delta &= 1/21 \\ \gamma &= 0.6 \\ \theta &= 0.01 \\ l &= 1 \\ \Delta &= 0.5\end{aligned}$$

In the constant price equilibrium prices and qualities are:

$$\begin{aligned}q^* &= 0.57 \\ p^* &= 1.28\end{aligned}\tag{51}$$

In the saw-tooth equilibrium prices and qualities are:

$$\begin{aligned}q_0 &= 0.6 \\ q_1 &= 0.12 \\ p_0 &= 1.29 \\ p_1 &= 1.27\end{aligned}$$

The price on odd days is 2% lower than on even days. From within a completely stationary environment prices cycle deterministically from high to low to high.

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Appendix

Proposition 1 *There exists a set of parameters that satisfy assumptions 1 through 5.*

Lemma 1 *If assumption 3 holds when $\Delta \rightarrow 0$ and $\gamma > \mu$ then assumption 1 holds when $\Delta \rightarrow 0$.*

Proof:

$$(\Delta \rightarrow 0) \Rightarrow a_0 \rightarrow l\theta\mu(1-\delta)[(1-\gamma)\delta + (1-\delta)\mu] - l\gamma\delta^2$$

which is greater than zero if

$$\gamma\delta^2 > \theta\mu(1-\delta)[(1-\gamma)\delta + (1-\delta)\mu] \quad (52)$$

$$(\Delta \rightarrow 0) \Rightarrow \tilde{a}_0 \rightarrow (2-\delta)l\{\theta\mu(1-\delta)[1-\delta\gamma] - l\gamma\delta^2\}$$

which is greater than zero if

$$\gamma\delta^2 > \theta\mu(1-\delta)[1-\delta\gamma]. \quad (53)$$

Inequality (53) is stronger than (52) if $\gamma > \mu$. □

Lemma 2 *There exists a set of parameters consistent with assumption 3 and 5 as $\Delta \rightarrow 0$.*

As $\Delta \rightarrow 1$ assumption 3 requires that

$$\theta < \frac{\gamma\delta^2}{\mu(1-\delta)(1-\delta\gamma)}. \quad (54)$$

As $\Delta \rightarrow 1$ assumption 5 tends to

$$\frac{\delta^2}{\mu(1-\delta)^2(2-\mu)} < \theta. \quad (55)$$

Therefore

$$\frac{\delta^2}{\mu(1-\delta)^2(2-\mu)} < \theta < \frac{\gamma\delta^2}{\mu(1-\delta)(1-\delta\gamma)}. \quad (56)$$

(56) is possible if:

$$\gamma > \frac{1}{1 + (1 - \delta)(1 - \mu)} \quad (57)$$

□

Note that (57) in the proof of lemma 2 is assumption 4. Hence assumption 4 is compatible with assumptions 3 and 5. From lemma 1 it then follows that there is a set of parameters consistent with assumptions 1, 3,4 and 5 where $\gamma > \mu$.

Finally, we show that assumption 2 is compatible. Assumption 2 is easier to satisfy the lower θ . Assumption 5 provides a lower bound on θ so must show it is possible to have:

$$\frac{\delta^2}{\mu(1 - \delta)^2(2 - \mu)} < \theta < 1 - \frac{\delta}{(1 - \delta)\mu} \quad (58)$$

which is true if δ is sufficiently small relative to μ .

□