DOI: 10.1111/joes.12508

ARTICLE

JOURNAL OF ECONOMIC SURVEYS WILEY

Is everything relative? A survey of the theory of matching tournaments

Ed Hopkins

School of Economics, University of Edinburgh, Edinburgh, UK

Correspondence
Ed Hopkins, School of Economics, University of Edinburgh, 30 Buccleuch Place, Edinburgh, UK.
Email: e.hopkins@ed.ac.uk

Funding information
Leverhulme Trust, Grant/Award Number: MRF-2020-072

Abstract
This paper surveys the theory of matching tournaments, and shows how they can be used to understand how the social environment influences economic decision making. Matching tournaments are games in which players choose efforts or investments before entering a matching market in which attractiveness depends on these investments. This results in rat-race-like competition even in large populations, inducing a concern for relative position and resulting in higher effort than in the absence of competition. However, when both sides of the market invest, the result surprisingly can be socially efficient. Applications and extensions are discussed.

KEYWORDS
externalities, games, inequality, matching, tournaments

JEL CLASSIFICATION
C72, D31, D62, D63, D81

1 | INTRODUCTION

> Two hikers are walking in the woods when they see a bear. One man bends down to tighten the laces on his shoes.
>
> The other man looks at him and says,"Are you crazy? You can't outrun a bear!"

> The first guy, while tying his shoes, replies, "I don't need to outrun the bear. I just need to outrun you."[1]

The joke works because this kind of relative competition is familiar to us all. But there is a tendency to think that this kind of rivalry is confined to situations involving a small number of agents, like the two hikers. As economists, we are familiar with the idea that as numbers become large, markets become competitive. So individuals do not have to act strategically and, for example, act as price takers not price setters. However, this survey considers a class of models where strategic considerations are just as important with an infinite population as with small numbers. This class, which we can call *matching tournaments*, have been used to model a number of social and behavioral phenomena such as marriage markets, social norms, and conspicuous consumption. But they also help us to understand fundamental issues about investment and efficiency.

The literature on matching tournaments starts with the classic paper by Cole et al. (1992). There a matching tournament is embedded as the stage game of a dynamic multigenerational model. But the crucial aspects are present: individuals choose investments before engaging in a marriage matching market and those investments influence attractiveness in matching. It is more than a tournament, because there is also matching, but it is not a classical matching problem, because some characteristics of the matchers are endogenous. Cole et al. show how different social arrangements or norms, such as marriage conventions, can influence material outcomes such as economic growth. Thus, from the start, matching tournaments have been linked with bringing a wider set of considerations into economics. In particular, they introduce a strong role for inequality both for outcomes and efficiency, all this without assuming any social preferences. Rather, the strategic nature of interactions in matching tournaments implies interpersonal externalities not present in standard competitive markets.

This survey starts with the basic matching tournament or "rat race" model from Cole et al. (1992). In this model, only one side of the matching process invests, while the other side chooses partners on the basis of those investments. In equilibrium, the investing side overinvests. Further work, including Hopkins and Kornienko (2004, 2010) and Hoppe et al. (2009), introduced a comparative static methodology based on stochastic orders. Using this, one can show that the level of investment increases further if inequality amongst matches increases and welfare falls, but these results are reversed when inequality in own endowments increases. This paper also analyzes an incomplete information model where rather than useful investments, agents engage in wasteful signaling to attract partners. This has been used to explain fashion, conspicuous consumption, and even the evolution of cooperation.

The survey also considers the two-sided investment model due to Peters and Siow (2002) who found, perhaps surprisingly, that, in this context, equilibrium investment can be efficient. This result is then contrasted with the noisy investment model of Bhaskar and Hopkins (2016), who find that unless the two sides are entirely symmetric, investment is too high. The survey concludes by addressing the question, which again integrates social arrangements with economic outcomes, as to whether an uneven sex ratio in the marriage market can lead to increased saving and investment.

The term "matching tournament" naturally arises because the model in question blends matching markets with tournaments. Multiprize tournaments are familiar from sports such as golf or athletics. There are a range of prizes that vary in value, with first prize (the most valuable) being awarded to the highest performer, second prize to the second highest, and so on. In economics, tournament models have been used to model internal and external labor markets, lobbying and other economic situations as surveyed in Konrad (2009).[2]

Matching problems are familiar from the work of Gale and Shapley (1962), Becker (1973), and Roth (1984). A group of agents seek to match with other individuals or with an institution, women match with men in marriages, students match with colleges, doctors with hospitals, and workers with a firm. Central to the analysis of this type of market is the concept of stability in the sense of Gale and Shapley (1962). A matching is stable if there is no blocking pair, that is a man and woman who would prefer matching with each other than with their assigned matches.

Matching tournaments borrow the idea of stability from matching problems, but make two major changes. First, traditionally, the attributes of participants in matching problems are assumed exogenous and fixed. Matching tournaments instead assume that there is a first stage in which participants can invest to make themselves more attractive to individuals on the other side of the matching market that takes place in the second stage. Second, preferences in classical matching problems are arbitrary, each individual can rank potential matches in any order. Matching tournaments rather assume a common ordering, everyone agrees who is the most attractive partner. For example, everyone agrees that whoever has the highest investment is the most attractive match. Importantly, given these conditions, there typically will be a unique stable matching, which is positive assortative, the most attractive individual on one side of the market will match with the most attractive partner from the other.

This creates a tournament-like structure with potential matches taking the place of prizes. Increasing one's effort, performance, or investment, keeping others' effort constant, will make one more attractive. Thus one attains a better match, just as in a tournament, a better performance can lead to a better prize. Or to put it another way, rather than a tournament organizer assigning prizes assortatively on the basis of performance, performance by increasing attractiveness leads to the same outcome arising endogenously from a matching market. Equilibrium will be similar to that in tournaments in that (at least when the equilibrium is in pure strategies) in equilibrium, the marginal cost of effort will equal the marginal expected increase in match/prize quality from increased effort. One further point is that in matching tournaments with two-sided investment, for example, both men and women can invest to make themselves more attractive, then the prize structure is itself endogenous.

Matching tournaments can involve either transferable (TU) or nontransferable utility (NTU). In the first case, transfers can be made so that potential partners can bargain over the terms of the match. For example, in a marriage market, the exact amount of dowry can be settled. I only cover the TU case very briefly in this paper, and refer the reader to papers such as Cole et al. (2001a) and Nöldeke and Samuelson (2015) for greater detail. In the NTU case in contrast, partners have to be accepted (or rejected) "as they are." One might think that this lack of ability to incentivize investment by side payments would lead to inefficiency. The tournament aspect to matching gives incentives for excessive investment, the public good aspect (investments benefit yourself and your future partner) gives an incentive for investment to be too low. Perhaps strangely, these can balance out and overall investment can be socially efficient, even under NTU, though perhaps only in symmetric situations.

A crucial feature of matching tournaments is that behavior is strategic even when the number of participants is infinite. Each competitor remains in particular competition with similarly ranked agents. For example, the strongest competitors are in competition for first place, the weakest to avoid last (just as the two hikers wanted to avoid the bear). Thus, this is very different from the case where many individuals compete for a single prize or match, where the least able have little chance of winning and thus their effort choice affecting their outcome. In a matching tournament, because there is a full range of possible matches or prizes, there is always something to compete over. In other words, there are what are sometimes called "positional externalities" (a term due to Frank, 1991). By occupying a place ahead of you, I force you to receive an inferior match. Thus, as first observed by Cole et al. (1992), this induces apparent social preferences in the form of competitive ordinal relative concerns. Equilibrium utility, even holding income or wealth constant, will be increasing in one's relative position.

An important consequence of this is that inequality matters. Increasing resources to one group can make a second group of individuals worse off even if there has been no direct change to the second group's own circumstances—just as one hiker getting shoes makes the other worse off. However, we will also see that the effect of inequality is complex. One insight from Hopkins and Kornienko (2010) is that, in matching tournaments, inequality in initial endowments and inequality in terms of potential matches or rewards both influence competitive incentives and welfare. However, they do so in opposite directions, with inequality in rewards increasing competition, but inequality in endowments decreasing it.

2 | THE BASIC RAT-RACE MODEL

In this section, I introduce the basic matching tournament model based on the original version introduced by Cole et al. (1992). The intention is to outline the simplest model that is representative of the general class and has interesting features. Specifically, the model has complete information, infinite populations, one-sided deterministic investment, and matching with NTU. Variations on these assumptions are considered later. Intuition is stressed over technical detail and simplicity over realism.

Let us consider a highly simplified model of a traditional society where marriage involves a payment by the family of the bride to that of the husband, a dowry. The decision about how much to invest in the dowry is made before entering the marriage matching market. These assumptions generate a simple two stage matching tournament. In the first stage, households make investment decisions, trading off the cost of investment versus the possible gain in terms of an improved match. How they fare in the marriage matching in the second stage depends on the investment choices of other households. This can be called a rat-race model as the main result is that in equilibrium the side of the market that invests, invests too much. All attempt to improve their relative position, but in equilibrium, all stay in the same place.

Households have either only one daughter ("female households") or one son ("male households"). There are a continuum of female households who have type s on $[0, 1]$ with strictly increasing differentiable distribution $F(s)$ with density $f(s)$. The possible interpretations of this type include an endowment of ability or wealth. An equal measure of male households have type t on $[0, 1]$ with strictly increasing differentiable distribution $G(t)$ with density $g(t)$. All households with a daughter have the same utility function,

$$U(x, s, t) = t(s - x), \tag{1}$$

which depends on the value of the match t, the cost of the investment x made, and one's own type s. For simplicity, households with a son make no investment decision. Their preferences can be represented by any increasing function $V(x)$. That is, they simply prefer their sons to be matched with a girl with the highest investment, in effect a dowry, available.

Solving the model backwards, I start with the second period when investment decisions have been made and marriage matching takes place. Matching is assumed to be measure preserving, that is, any matching scheme must match an equal measures of males and females (this is roughly equivalent to one-to-one matching with finite numbers, or simply, each marriage is between one female and one male). I show that, given male households prefer higher investment, equilibrium matching will then be positive assortative in investment. That is, the higher the investment by a female household, the higher the type of the son their daughter will marry.

The next crucial assumption is that marriage matching will be *stable* in the sense of Gale and Shapley (1962). That is, a proposed matching is stable if there is no "blocking pair," a man and woman who would prefer to marry each other rather than the matches they would have under the proposed matching. Second, suppose further that investment x is smoothly and strictly increasing in initial wealth s (Proposition 1 shows that this holds in the unique equilibrium). This implies that if we aggregate the investment decisions of the female households into an (endogenous) distribution function $\tilde{F}(x)$ that describes the distribution of investment amongst the female households, then $\tilde{F}(x)$ is smooth and strictly increasing.

One can then prove that the only stable matching is positive assortative matching (PAM) where each female household i matches with a husband whose wealth t_i has exactly the same rank in the distribution of men's wealth as the female household has in investment.[3] This follows directly from assuming monotone preferences on each side. To see this, the woman who has the highest dowry can confidently propose to the wealthiest man as she knows she is his preferred match, since men's preferences are simply increasing in dowry size. Similarly, a low-ranked man can see that, while he would like to marry a woman with a large dowry, she would prefer a man of higher type to him.

Formally PAM, given that investment is strictly increasing in type s, implies that

$$\tilde{F}(x_i) = F(s_i) = G(t_i). \tag{2}$$

That is, household i's investment x_i has the same rank in investment as its initial type s_i held in the initial distribution $F(s)$. The daughter then matches with a son of type t_i who has the same rank $G(t_i)$ in the distribution of male households. The structure of this PAM is illustrated in Figure 1. Given PAM, in theory, this daughter could calculate the type of her future husband by inverting the male household distribution function G to obtain $t_i = G^{-1}(F(s_i))$. More generally, one can define the matching function $T(s)$, which will be heavily used in subsequent analysis, as

$$T(s) := G^{-1}(F(s)). \tag{3}$$

Note that its derivative is $T'(s) = f(s)/g(T(s))$. As we will see, this implies that the marginal benefit of investment through better matching will be increasing in own density $f(s)$ but decreasing in the other side's density $g(t)$.

Given this unique stable matching, it is now possible to move backward to the investment stage of the model. First, suppose PAM was assigned by a central planner, rather than determined by endogenous investment choice. Then, what level of investment would households choose? This level of investment that is optimal in the absence of matching considerations will be useful as a point of comparison with the Nash equilibrium level of investment that will eventually be derived. Here, because x does not enter directly enter into household utility $U(x, s, t)$, the privately optimal level of investment is $x_p(s) = 0$ for all s.

Next, turning to the noncooperative equilibrium, female households anticipate PAM when they choose investments, taking into account that increased investment will raise the quality of match achieved. For example, in Figure 1, if the household raises x in the top left segment, it will have higher induced rank $\tilde{F}(x)$ in the distribution of investments, leading to a higher match value

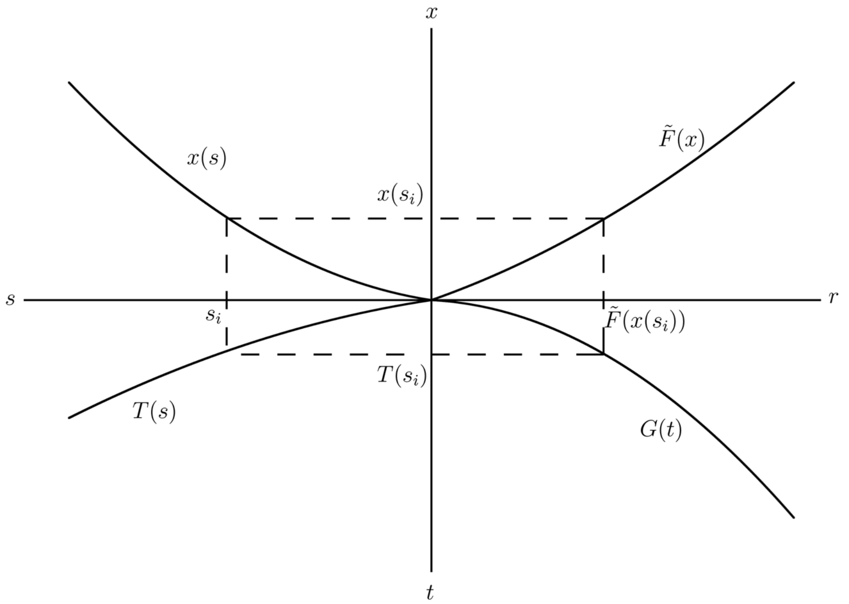

FIGURE 1 Investment and matching: a female household that has type s_i, has investment $x(s_i)$, holds rank $r = \tilde{F}(x(s_i))$ in the distribution of investment, and matches with a male household of the same rank in the distribution $G(t)$ and thus gains a match of value $t_i = T(s_i) = G^{-1}(\tilde{F}(x(s_i))) = G^{-1}(F(s_i))$

t. Formally, if all female households use a strictly increasing investment strategy $x(s)$, then a female household anticipates utility $U(x, s, T(s))$ in equilibrium, with the rank-preserving matching function $T(s)$ replacing t in the household utility function (1). Note that utility, through $T(s)$, now depends on both the distribution $F(s)$ of female households' wealth and the distribution $G(t)$ of male households.

Let us now look for the equilibrium investment schedule $x(s)$. Suppose given some candidate equilibrium $x(s)$, one agent of type s considers a deviation to investing $x(\hat{s})$ instead. Her reduced form utility is $U(x(\hat{s}), s, T(\hat{s})) = T(\hat{s})(s - x(\hat{s}))$. Equilibrium will hold if such a deviation is not profitable. We check this by supposing she then chooses $\hat{s}$ to maximize her payoff.[4] Then, differentiating with respect to $\hat{s}$ gives a first order condition,

$$x'(\hat{s})T(\hat{s}) = T'(\hat{s})(s - x). \tag{4}$$

The left hand side gives the marginal cost of increased investment, the right hand side, the marginal benefit in terms of the improved match. Now, in a symmetric equilibrium, it must be that $\hat{s} = s$. Using this and rearranging the resulting first order condition, we have the following differential equation:

$$x'(s) = \frac{T'(s)}{T(s)}(s - x). \tag{5}$$

This differential equation has the boundary condition $x(0) = x_p(0) = 0$. That is, the lowest ranked household acts as though matching considerations do not matter. This is the best response to the fact that in equilibrium, the lowest type comes last and is matched with the lowest type on the other side for sure. Thus, there is no point investing more. One can write Equation (5) as $x'(s)T(s) + x(s)T'(s) = T'(s)s$. Then integrating both sides in combination with the boundary condition gives the explicit solution,

$$x(s) = s - \frac{\int_0^s T(u)\,du}{T(s)} = \frac{\int_0^s u\,dT(u)}{T(s)}. \tag{6}$$

One can show that this equilibrium is unique.

Proposition 1. *There is a unique symmetric strictly increasing equilibrium to the matching tournament given by investment function $x(s)$, which solves the differential equation (5) with initial condition $x(0) = 0$. This equilibrium is Pareto dominated for female households by PAM with zero investment $x(s) = 0$.*

The next task is to explain why this equilibrium is inefficient from the point of view of the female households. As a point of comparison, remember that if the central planner imposes PAM, then the optimal investment is $x_p(s) = 0$ for all s. Then, given x is everywhere zero, female households have utility $U_p(s) = T(s)s$. Whereas in the noncooperative equilibrium, households have utility $U(x(s), s, T(s)) = U(s)$, which given the equilibrium strategy given above, will be

$$U(s) = T(s)(s - x(s)) = \int_0^s T(u)\,du = T(s)s - \int_0^s u\,dT(u) \le T(s)s. \tag{7}$$

That is, every female household earns less in the noncooperative equilibrium than if all invested what is privately optimal but still were matched assortatively. This rat-race result dates back to Frank (1985). The utility specification there (and in Hopkins and Kornienko (2004, 2009, 2010)) is more general, allowing for a private return to investment such as for example $U(x, s, t) = v(x, s - x)t$ for some increasing concave function $v(\cdot)$.[5] While this means that the privately optimal investment is greater than zero, investment in the noncooperative equilibrium will be even higher and thus the equilibrium is inefficient. In Cole et al. (1992), the matching tournament is part of a dynamic economy, so that higher investment leads to higher growth than when investment is privately optimal.

However, one should note that these inefficiency results are literally one-sided. In this marriage matching example, the benefit that male households receive from the additional investment expenditure is not considered. In Frank's (1985) original rat-race result, there was no other side. Rather, he assumed an intrinsic desire for relative position or status. When there are male households that benefit from investment, equilibrium investment could even maximize overall welfare, despite being excessive from a female viewpoint. We will obtain a clearer picture of the welfare issues involved, when we look at incomplete information Section 2.2 and at two-sided investment, Section 3.

For now, we can discuss some other important aspects of the basic model. One question might be about how does the actual matching procedure take place, to which there are two answers. The first, familiar from cooperative game theory, is simply to assume that the matching market is frictionless and so only the efficient outcome, the unique stable matching, can emerge. Second, if one prefers a noncooperative approach, Cole et al. (1998) outline a specific procedure, where each man is allowed to make exactly one marriage proposal to a woman. Women accept their most preferred offer if they receive multiple offers, and accept their only offer if they receive only one. Then the equilibrium outcome is the unique stable matching.

Another important point to make is that this matching tournament is strategically similar to a *two bidder* first price auction. In a first price auction, a bidder trades off the increased probability of winning with the increased cost of a higher bid. Given the standard symmetric equilibrium, the probability of winning in the two bidder first price auction is equal to one's rank $F(s)$ is the distribution of types. Similarly here utility is increasing in $T(s) = G^{-1}(F(s))$ and participants must trade off rank in the field of competitors with the cost of investment. But in the auction as the number n of bidders increases, the probability of winning $F^{n-1}(s)$ approaches zero, as many bidders compete for the single object for sale. In contrast, in the matching tournament where the number of potential partners is as large as the number of competitors, the tournament is as strategic as the auction with only two bidders.

Finally, this rat-race model also gives a justification for a type of social preferences. For example, in the famous paper of Frank (1985), he assumes that individuals have an intrinsic interest in their relative position. Here, in contrast, the rat-race model makes the standard neoclassical assumption that households only care about their own material payoffs. Nonetheless, note that one can write equilibrium utility (7) as

$$U(x, s, T(s)) = \int_0^s T(u)\, du = G^{-1}(r).s - \int_0^s u\, dT(u)$$

where a household's rank r is defined as $r = F(s)$, its position in the distribution of types. That is, the tournament with assortative award of prizes implies that each individual's payoffs are increasing in her rank r in the distribution of contestants. It therefore might appear to an outside observer that the individual had some form of social preferences where she cares about her relative position, similar to those analyzed by Frank (1985) and Hopkins and Kornienko (2004). As Cole, Mailath and Postlewaite (1992) and Postlewaite (1998) point out, this form of tournament, therefore, gives an instrumental economic basis to such models.

2.1 | Which inequality?

If we now consider the impact of inequality in this rat-race model, it also seems consistent with a form of social preferences. An individual can become better or worse off without any change to her material circumstance but because others' situations change. Importantly, as Hopkins and Kornienko (2010) argue, here there are two very different types of inequality, inequality in endowments $F(s)$, and inequality in matches or rewards $G(t)$. An increase in equality in endowments (perhaps unexpectedly) increases the rat-race effect and lowers welfare, but increased equality of rewards decreases equilibrium investment and thus increases the welfare of the investing households.

As a first step, note that the distribution of endowments $F(s)$ and the distribution of matches $G(t)$ both directly influence the equilibrium strategy (6) and utility (7). One way of understanding (6) is that investment by an individual of type s is increasing in the average endowments of those with type lower than s. The individual maintains his position in the tournament by resisting pressure from below. An increase in equality in endowments in the sense of second-order stochastic dominance (SOSD) will redistribute resources to the lower end of the distribution. Thus, competitive pressure increases, investment rises and utility falls. This is illustrated in Figure 2.

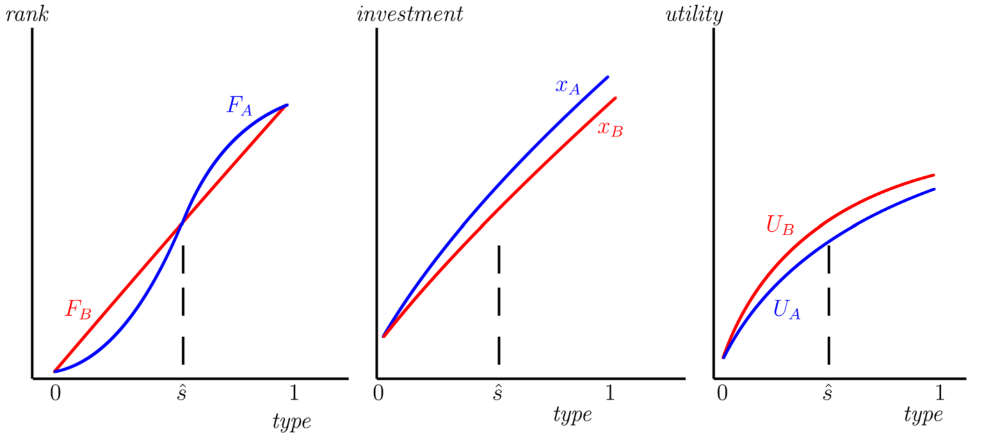

FIGURE 2 Illustration of Proposition 2 (a): Greater equality in endowments under income distribution F_A (first panel) leads to higher investment for all (second panel) and lower utility for all (third panel) [Colour figure can be viewed at wileyonlinelibrary.com]

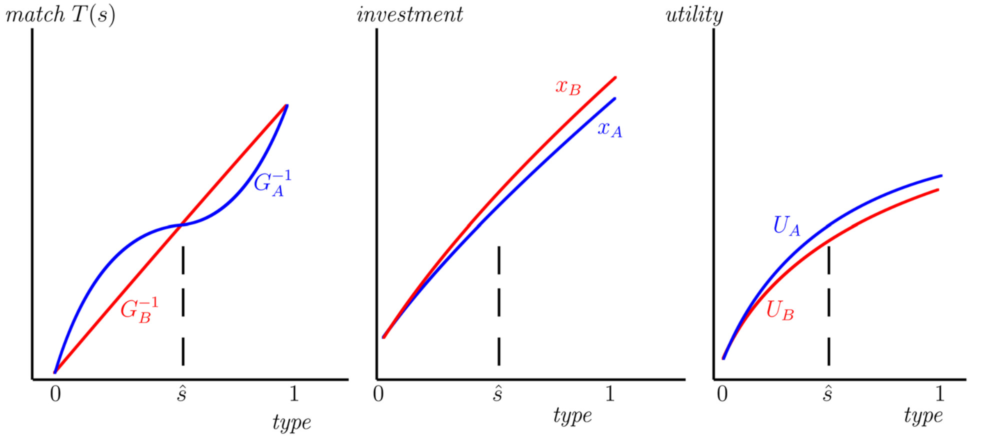

FIGURE 3 Illustration of Proposition 2 (b): Greater equality in matches under inverse match distribution G_A^{-1} (first panel) leads to lower investment for all (second panel) and higher utility for all (third panel) [Colour figure can be viewed at wileyonlinelibrary.com]

However, as Hopkins and Kornienko (2010) point out, the reverse happens with an increase in the equality of the match distribution. An increase in reward equality reduces the incentive to invest, which diminishes the intensity of the rat-race, raising welfare. This is illustrated in Figure 3.

Proposition 2. *(a) Fix $G(t) = t$, a uniform distribution. Then, consider two endowment distributions F_A, F_B with the same means, but $F_A \succ_{SOSD} F_B$. Then under the more equal distribution, F_A, at every endowment level $s \in (\underline{s}, \bar{s})$ investment is higher and utility is lower. (b) Fix $F(s) = s$, a uniform distribution. Then, consider two match distributions G_A, G_B with the same means, but $G_A \succ_{SOSD} G_B$. Then under the more equal distribution, G_A, at every endowment level $s \in (\underline{s}, \bar{s})$ investment is lower and utility is higher.*

To expand on the intuition for this result, consider a foot race. The fastest runner should win, but how fast she has to run to do so depends on how many almost as quick runners are in the field. In a matching tournament, the equilibrium is monotone, the highest type always gains the best match. However, greater equality in types means more competitors of similar ability and thus greater competition, leading to a greater dissipation of resources in investment. In contrast, a decrease in the dispersion of rewards or prizes for the race decreases competitive incentives, reducing dissipation.

There is an important qualification to the result Proposition 2(a). One might think that it implies that greater inequality would result in a Pareto improvement. This is not the case. The result is that utility is higher under distribution $F_B(s)$ at any fixed endowment level s. However, to change the distribution from $F_A(s)$ to $F_B(s)$ necessarily the distribution of endowments has changed, so that a large proportion of the population do not stay at a fixed endowment level. So, as Hopkins and Kornienko (2009) point out, a fairer comparison may be at constant ranks, not constant endowments, which thus includes changes in endowments in analyzing the effect on welfare. In particular, greater inequality will typically reduce endowments of the low-ranked. Thus, overall they are likely to be worse off, and thus this change does not result in a Pareto improvement.

2.2 | Incomplete information and signaling

For a number of applications, an incomplete information signaling approach is more appropriate. Rather than making an investment that increases attractiveness, agents choose a visible signal that reveals information about their underlying type. This type is correlated with attractiveness to the other population. An example of this is conspicuous consumption—an individual wears expensive items to signal her high underlying wealth. A positive assortative equilibrium is possible where high types send high signals and match better than low types. Early work in this direction includes Cole et al. (1995) and Pesendorfer (1995). Further papers include Rege (2008), Hoppe et al. (2009), Hopkins and Kornienko (2010), Hopkins (2012), and Coles et al. (2013).

As Hoppe et al. (2009) and Hopkins (2012) find, signaling in matching markets works quite differently from the original Spence signaling model. Here, signals, by revealing an agent's type, determine her match or job assignment. This affects total productivity and welfare (see Proposition 3 below), whereas total productivity in the Spence model is fixed. Further, while in the original Spence model, equilibrium signaling is unaffected by relative frequencies of types, here equilibrium outcomes are affected, as in matching tournaments in general, by the distributions on both sides. So, here, if the quality or quantity on the signaling side of the tournament increases, signaling should increase.

Just as in the complete information model, assume all households with a daughter have the same utility function, $U(x, s, t) = t(s - x)$, which as before depends on the value of the match t, and one's own type s, but now, x is the cost of the signal made. The crucial difference, however, is that households with a son have preferences represented by an increasing function $V(s)$. That is, they now care about the daughter's underlying type not about investment.[6]

Perhaps surprisingly, in this new model, there is a monotone, separating equilibrium that is identical to the equilibrium under complete information. That is, the unique separating equilibrium for this signaling model is given by the strictly increasing strategy $x(s)$ that solves the differential equation (5) with initial condition $x(0) = 0$. Given that in a separating equilibrium types are revealed, a female household faces the same marginal trade-off between the cost of increasing expenditure against gaining a better match.

Suppose indeed that all female households employ a strictly increasing strategy $x(s)$, then male households will be able to infer the relative ranking of female households—because each female household has the same ranking in signals as its ranking in underlying type. This implies that the only stable matching is positive assortative just as in the complete information model of the previous section. Thus, the female households have the same matching function $T(s) = G^{-1}(F(s))$. Then, an individual female household considering deviation from this strategy to $x(\hat{s})$ expects a match $t = T(\hat{s})$. So, her utility would be of the form $U(x, s, T(\hat{s}))$, the payoff to the signaller depends on her action, her type, and her perceived type. One can then apply the results of Mailath (1987) to show that indeed there is a unique separating equilibrium satisfying the differential equation (5) and thus having the exact same solution (6). Thus, there is a crucial difference from the classic Spence signaling model. Here, the separating equilibrium depends on both type distributions, the distribution of those signaling and those receiving and will respond in the same way as identified in the comparative statics results of Section 2.1.

Importantly, in contrast to the situation under complete information, this is not the only equilibrium. While there is only one separating equilibrium, there are other equilibria. In particular, there is a pooling equilibrium in which no-one sends any signals so that $x(s) = 0$ for all types. Since there is no information transmitted, in the second stage, all matching is completely random. Thus, a female household of type s gains utility $\int st\, dG(t) = s\bar{t}$, where $\bar{t}$ is average male type. This can be maintained as an equilibrium by the male households having beliefs that any female household that sends a signal $x > 0$ has a type s that is lower than average. Such an equilibrium of course does not exist in the investment model of the previous section because there a higher investment automatically makes a female more attractive. Again contrast the current preferences for the receivers $V(s)$ with the preferences $V(x)$ in the complete information model.

In any case, this raises the question which equilibrium generates higher welfare. Given the multiplicative form of the utility (1), the PAM in the separating equilibrium maximizes total payoffs for a given cost x.[7] However, in the separating equilibrium, in aggregate, costs can exceed the benefits from more efficient matching. The pooling equilibrium can in fact generate higher total welfare by avoiding any signaling costs.

Specifically, at the individual level, high types on the signaling side of the market always prefer separation because this brings them high-quality matches. In contrast, low types prefer pooling as a random match is higher in expectation than the low match they can get under assortative matching. Overall, Hoppe et al. (2009), using earlier results by Barlow and Proschan (1965) on the properties of distributions, find that total welfare is only higher in the separating equilibrium if the distribution of types is highly dispersed, that is, if very high types are relatively common. Then PAM, by matching them with other high types, results in a high level of output, raising average welfare. But if not, the costs of maintaining a separating equilibrium are higher than the gain in output.

Proposition 3. *Let $F = G$. Then, assortative matching based on signaling is welfare superior to random matching for the signaling population if and only if the common coefficient of variation ($CV(s) = \sqrt{Var[s]}/E[s]$) is greater than 1.*

The analysis for the receiving side is similar but simpler. At the individual level, the payoffs for an individual of type t are $V(t)$ in the separating equilibrium, and $V(E[t])$ in the pooling equilibrium. Again high types ($t > E[t]$) prefer separation. But average welfare is higher under pooling if V is concave. What if both sides signal? Hoppe et al. (2009) find that again the separating equilibrium generates higher total welfare only if the type distributions are sufficiently dispersed.

3 | TWO-SIDED INVESTMENT

We now modify the model to allow both sides of the matching market to invest. This adds greater realism. Further, as argued above, a full treatment of both sides is necessary to make a satisfactory analysis of the efficiency of investment decisions. The literature on two-sided investment divides between NTU and TU assumptions on matching, with Peters and Siow (2002) pioneering the former and with Cole et al. (2001) and Felli and Roberts (2000) pioneering the latter.

The main finding here is that investment can be efficient, thus avoiding a potential "hold-up problem." This is surprising because one side's investment will benefit the other side, but there is no reason for the investing side to internalize this benefit. For this reason, one would thus expect investment to be under-provided in a noncooperative equilibrium. However, as in the one-sided model of the previous section, competition for matches will drive up investment beyond what is privately optimal. So, this factor might lead to excessive investment. Further, in contrast to the TU case, there is no way of compensating the investor through a suitable side payment. Thus, the fact that these two different factors exactly cancel each other out is very surprising.

The set-up is similar to that in Section 2. There are a unit mass of women and of men. Women have a type s distributed according to $F(s)$ on $[\underline{s}, \bar{s}]$ and men have type t distributed according to $G(t)$ on $[\underline{t}, \bar{t}]$. However, now both female and male households make investment decisions. If a woman makes an investment x and matches with a man with investment y, the payoff to a woman, respectively, man, will be

$$U(x, y, s) = x + y - c(x, s); \; V(x, y, t) = x + y - c(y, t), \tag{8}$$

where the cost function c is increasing in investment and decreasing in one's type, formally $c_1 > 0$, $c_2 < 0$, $c_{12} < 0$. That is, we keep Peters and Siow's (2002) quasilinear specification.[8]

Note that investments here operate as a local public good. Both the man and woman gain equally from the sum of investments. In the next section, we will see an alternative assumption that investments are divisible, leading to a TU setting.

The first question is what would be the socially efficient investments in this case. Take a social welfare function that places a weight $\lambda \in (0,1)$ on women and $1-\lambda$ on men,

$$W = \lambda U(x, y, s) + (1-\lambda)V(x, y, t) \tag{9}$$

for some $\lambda \in (0,1)$. Conditions for Pareto efficiency are then

$$\frac{1}{\lambda} = c_x(x, s); \frac{1}{1-\lambda} = c_y(y, t). \tag{10}$$

The condition for utilitarian efficiency (i.e., $\lambda = 1 - \lambda = \frac{1}{2}$) is, therefore, $c_x(x, s) = c_y(y, t) = 2$. But note that by combining the above conditions for Pareto efficiency, one obtains

$$c_y(y, t) = \frac{c_x(x, s)}{c_x(x, s) - 1} \Leftrightarrow c_x(x, s) = \frac{c_y(y, t)}{c_y(y, t) - 1}. \tag{11}$$

This relation holds irrespective of the value of λ. In contrast, what is privately efficient solves $c_x(x, s) = 1 < 1/\lambda$ for women and $c_y(y, t) = 1 < 1/(1-\lambda)$ for men. That is, privately optimal investments are below the socially efficient levels, quite simply because in the privately optimal decision an individual does not take into account the benefit of her investment for her future partner.

Matching will be positive assortative in terms of chosen investment, with the highest investing woman matching with the highest investing man and so on down. Assume that women's investment $x(s)$ is strictly increasing, as is $y(t)$ the men's investment function. Then, a woman of type s who chooses investment as though she was type $\hat{s}$ would have expected utility,

$$U = x(\hat{s}) + y(T(\hat{s})) - c(x(\hat{s}), s). \tag{12}$$

For a man of type t, assume he chooses to bid as if he was type $\hat{t} = T(\hat{s})$. Then, his expected utility can be written

$$V = x(\hat{s}) + y(T(\hat{s})) - c(y(T(\hat{s})), t). \tag{13}$$

Differentiating each with respect to $\hat{s}$, together they give the first-order conditions,

$$x'(s) + y'(T(s))T'(s) - c_x(x(s), s)x'(s) = 0 \tag{14}$$

$$x'(s) + y'(T(s))T'(s) - c_y(y(T(s)), t)y'(T(s))T'(s) = 0.$$

Combining them one has

$$x'(s)\left[\frac{c_y(y,t)}{c_y(y,t)-1} - c_x(x,s)\right] = 0. \tag{15}$$

One can see that the Pareto efficiency condition (11) must be satisfied by any equilibrium in increasing strategies that satisfies this first-order condition.

Proposition 4. *In the double-sided investment game with NTU, all monotone equilibria are Pareto-efficient.*

But why do we get efficiency? The first-order condition for women's investment can be written as $1 - c_x + y'T'/x' = 0$. The component $1 - c_x$ is the privately optimal incentive to invest, the other $y'T'/x'$ is the incentive to invest to gain a better match. To match the socially optimal incentive to invest as given in Equation (11), then this additional incentive to invest has to equal the additional benefit of that investment to the woman's match. Now, suppose the two genders in this market are completely symmetric with equal distributions of types, $F(\cdot) = G(\cdot)$ so that $T(s) = s$ and $T'(s) = 1$. Then, given symmetry, the two sides will use the same strategies in equilibrium so that $y'(T(s)) = x'(s)$, so that $y'T'/x' = 1$. Then the first-order condition for women will be $2 - c_x = 0$, which is equivalent to the social optimum first-order conditions with $\lambda = \frac{1}{2}$. That is, in a symmetric matching tournament, the additional incentive to invest is equal to the social benefit with symmetric Pareto weights.

What if the situation is not symmetric? Clearly, then T' will not equal one, nor will $x' = y'$, so that $y'T'/x'$ will not equal one. However, if one compares Equation (10) with Equation (14), then they are equivalent if $y'T'/x' = 1/\lambda - 1$. One can simply find a λ such that this is the case—that is, $\lambda = 1/(1 + y'T'/x') \in (0,\ 1)$. One can check that this is consistent with having a Pareto weight of $1 - \lambda$ for the other gender. That is, when the situation is not symmetric, investments will not be symmetric. This will still be efficient but consistent with an asymmetric Pareto weight.

To see how this works in practice, one can construct some specific examples. Suppose that $c(x, s) = -\ln(s - x)$, $c(y, t) = -\ln(t - y)$. The efficiency condition becomes $x + y = T(s) + s - 1$. Differentiating this efficiency condition with respect to s, one obtains

$$x'(s) + y'(T(s))T'(s) = 1 + T'(s). \tag{16}$$

Combined with the first-order condition (14), this results in an ordinary differential equation for the exact investment strategy,

$$x'(s) = (1 + T'(s))(s - x). \tag{17}$$

However, in contrast to the model of the previous section, there is no obvious initial condition and so there are many solutions. Each will be efficient but involve a different distribution of payoffs between the two sides. Put another way, the equilibrium investments must be on the contract curve between the marrying parties, but it does not say which exact point.

Example: Let $F(s) = s - 2$ and $G(t) = t - 2$ on $[2,\ 3]$ so that $T(s) = s$ and $T'(s) = 1$. Then one can calculate that $x'(s) = 2(s - x)$ and therefore, $x(s) = s - 1/2 + e^4 e^{-2s}(\underline{x} - 3/2)$, where $\underline{x} = x(2)$, the investment choice of the lowest ranked woman. If this woman chooses the utilitarian efficient investment (which is symmetric between the genders) $\underline{x} = 3/2$ then the equilibrium is the simple linear function $s - 1/2$. But there are also an infinite number of other solutions depending on the exact value of $\underline{x}$. The efficiency condition at the bottom is $\underline{x} + \underline{y} = T(\underline{s}) + \underline{s} - 1 = 3$ and $\underline{x}$ can be as high as 1.865 (which makes the lowest woman indifferent between this investment pair which gives utility $3 + \ln(2 - 1.865) = 1$ and being unmatched with her privately optimal investment $x = 1$, utility $1 + \ln(2 - 1) = 1$) or as low as 1.135 (which makes the lowest man indifferent).

If we now change the men's distribution of types to be $G(t) = 2t - 4$ uniform on $[2, 2\frac{1}{2}]$, more compressed, then $T(s) = (s + 2)/2$, $T'(s) = \frac{1}{2}$, $x'(s) = \frac{3}{2}(s - x)$, and $x(s) = s - 2/3 + e^3 e^{-3s/2}(\underline{x} - 4/3)$. Now the simple equilibrium is $x(s) = s - 2/3$, women invest less than in the previous example. This is still efficient and is consistent with a Pareto weight of $\lambda = 2/3$, more weight placed on the welfare of women. However, this arises in this noncooperative setting because men face greater competition because their distribution is more compressed. The effect is similar to that in the first part of Proposition 2.

This efficiency result is both surprising and important but there are two major qualifications. First, as already noted, there is an infinite number of efficient equilibria, as the first-order condition only ties down the slope of the solution, but the boundary condition is not unique. Second, as Bhaskar and Hopkins (2016) point out, there is an unanswered question about investment below the equilibrium support. Specifically, if an individual deviates to an investment below the minimum level for her side, what match does she receive? Given equilibrium investment functions $x(s)$, $y(t)$ and the standard matching function $T(s)$, we can define $y(x) = y(T(s^{-1}(x)))$, the value of the investment of one's match y, given one's own investment x. Given the equilibrium, this function is only clearly defined on $[x(\underline{s}), x(\bar{s})]$. There are two prominent possibilities for the return to investments $x < x(\underline{s})$that are below the lower bound. The first is that one is matched with the lowest type on the other side so that $y(x) = y(\underline{t})$for $x < x(\underline{s})$; the second is that one becomes unmatched, $y(x) = 0$.

There is a major problem with the first option in that it leads to the equilibrium unraveling. Consider the lowest type on either side. If he deviates downwards to his privately optimal investment, then he receives the same match as before and so such a deviation is profitable. But if this lowest type invests what is privately optimal then this investment is inefficient and so we are no longer in the monotone equilibrium which we saw must be efficient. Thus, to maintain equilibrium, one needs the second option (or similar) so that even the bottom agents have no incentive to deviate down from the efficient equilibrium investments. Take the first example above, the lowest ranked female $s = 2$ matches with a man of $t = 2$ and get utility $3 + \ln(1/2)$ in the equilibrium where they both invest 3/2. If she deviates downwards to her privately optimal investment $x = 1$, instead of receiving $y = 3/2$, she gets nothing and overall obtains $1 + \ln(1)$, which is lower.

The problem with the second option is that, first, the threat of being unmatched does not seem to be credible in the sense of subgame perfection. For example, the lowest ranked man has, in effect, to threaten not to marry the lowest ranked women if she reduces her investment, even at the cost of remaining unmarried himself. Second, given the assumption that downward deviators are not matched, there are also many other non-monotone inefficient equilibria. For example, consider pooling equilibrium where investment levels are constant and irrespective of one's type. Carrying on the above specific utility and type distribution example, the investment functions $x(s) = y(t) = 3/2$ also form an equilibrium. What makes this an equilibrium is the assumption that if, for example, a seller deviates downwards, he will be unmatched. Thus, the payoff in the proposed equilibrium will be for a woman $3 + \ln(s - 3/2)$. But if she deviates to her privately optimal investment, $x = s - 1$, she obtains a match value of $y = 0$ and overall payoff $s - 1 + \ln(1)$, which is less.

It seems that given this threat—invest as prescribed or remain unmatched—anything could be an equilibrium. This is not true, at least in the sense that any strictly monotone equilibrium must be efficient. For example, everyone investing what is privately optimal, here $x = s - 1$ and $y = t - 1$, is not an equilibrium. The point is that once the other side of the market invests so that one faces a range of possible matches, one has an incentive to invest to improve one's match. Then to solve the first-order conditions (14), the investment function must be efficient.

3.1 | Noisy investment

We have seen that with double-sided investment, we have the surprising result that the outcome can be efficient. However, an important qualification is the presence of multiple equilibria, not of all of which are efficient, and underlying this, the equilibrium concept itself. As already noted above, it is in effect not subgame perfect, in that downward deviations are deterred by threats of being unmatched, which are not clearly motivated. The aim of Bhaskar and Hopkins (2016) (BH16) was to provide a better foundation. They assume that investment has a stochastic outcome so that a wide range of investments, including downward deviations, will be on the equilibrium path.[9] They find that in this setting, efficient investment is still possible but only if the two genders are identical. If not, for example, if they differ in costs, distribution of outcomes, size of population or otherwise, then total investment is too high.

Take the setting of Section 3 but now assume that all women have the same type s and men have the same type t. However, the principal innovation is that the outcome of investment is stochastic:

a woman who invests x has a final realization of that investment equal to $q(x,\eta) = x + \eta$, where η is a random shock. The shock is independent across individuals and is distributed continuously on $[\underline{\eta}, \bar{\eta}]$ with distribution function $F(\eta)$. For men, ex post investment is $q(y,\varepsilon) = y + \varepsilon$, where ε is distributed according to $G(\varepsilon)$. Agents must choose investments before the shock is realized, but then match on the basis of the final investment.

So, a woman who invests x and matches with a man of quality $q(y,\varepsilon)$ has utility

$$U(x,y) = q(y,\varepsilon) + q(x,\eta) - \tilde{c}(x) = q(y,\varepsilon) - c(x). \tag{18}$$

That is, we combine the private benefit $q(x,\eta)$and cost $\tilde{c}$into a net cost function $c(x)$ (dependence of costs on type s is dropped as there is no heterogeneity here). And a man investing y matched with a woman with $q(x,\eta)$ will obtain

$$V(x,y) = q(x,\eta) + q(y,\varepsilon) - \tilde{c}(y) = q(x,\eta) - c(y). \tag{19}$$

The crucial aspect is that utility is increasing in the ex post investment value q of one's match.

There will be a pure strategy Nash equilibria in which every agent on a given side of the marriage market chooses the same level of investment. Call such an equilibrium quasi-symmetric and denote the equilibrium investments (x^*, y^*).

In such an equilibrium, because all women invest the same amount, the ex post distribution of qualities will be given by the distribution of shocks $F(\eta)$. Thus, equilibrium matching will be positive assortative with the matching function given by

$$\phi(\eta) := G^{-1}(F(\eta)). \tag{20}$$

This is conceptually the same as the matching function $T(s)$ (introduced in Equation (3)), but in terms of distributions of shocks not types. Its derivative is similarly the ratio of the densities of the two distributions, $\phi'(\eta) = f(\eta)/g(\phi(\eta))$.

What will be the conditions for a quasi-symmetric equilibrium? Consider a woman who is thinking of deviating from the common investment level $x^*$to $x^* + \Delta$. If her shock realization is η, then her new ranking within women will be $F(\eta + \Delta)$. Her new match will have rank $G(\phi(\eta + \Delta))$ and quality $y^* + \phi(\eta + \Delta)$. Taking expectations across all possible shock values, the total expected return to an upward deviation of Δ would be,

$$U(x^* + \Delta, y^*) = \int_{\underline{\eta}}^{\bar{\eta}-\Delta} (y^* + \phi(\eta + \Delta)) f(\eta)\, d\eta + (1 - F(\bar{\eta} - \Delta))(y^* + \bar{\varepsilon}) - c(x^* + \Delta). \tag{21}$$

Thus, differentiating with respect to Δ and setting to zero, female and male households face the following respective first-order conditions

$$\int_{\underline{\eta}}^{\bar{\eta}} \frac{f(\eta)}{g(\phi(\eta))} f(\eta)\, d\eta = c'(x^*); \quad \int_{\underline{\varepsilon}}^{\bar{\varepsilon}} \frac{g(\varepsilon)}{f(\phi^{-1}(\varepsilon))} g(\varepsilon)\, d\varepsilon = c'(y^*). \tag{22}$$

On the right-hand side is the marginal cost of increased investment, on the left, the marginal benefit in terms of a better match. As before, this benefit is proportional to the slope of the matching function $\phi'(\eta) = f(\eta)/g(\phi(\eta))$. However, now the individual takes expectations over the range of possible outcomes of the shock she faces because the investment decision is taken before the realization of the shock.

The final question is efficiency. As in the previous section, one can consider utilitarian or more general Pareto efficiency criteria. Placing a weight λ on U as given in Equation (18) and $1-\lambda$ on V as in Equation (19), the latter will be

$$c'(x) = \frac{\lambda}{1-\lambda};\ c'(y) = \frac{1-\lambda}{\lambda} \tag{23}$$

This implies an important further condition $c'(x) \times c'(y) = 1$. The stronger utilitarian condition is $c'(x) = c'(y) = 1$.

Proposition 5. *There exists a unique quasi-symmetric Nash equilibrium of the matching tournament. In this quasi-symmetric equilibrium investments are generically excessive relative to Pareto efficiency.*

One reason for uniqueness is that (small) downward deviations will be on the equilibrium path given that investment has a noisy return. Thus, the model avoids some of the problems raised in the previous section in the context of deterministic investment. However, BH16 also employ an argument based on a model with finite numbers to ensure that larger downward deviations are not profitable.

Example 1. Here is a specific example of excessive investment. Let us assume that $F(\eta) = \eta$ on $[0, 1]$, that is, η is uniformly distributed. Assume that $G(\varepsilon) = \varepsilon^n$ on $[0, 1]$. $F(\varepsilon) = G(\phi(\eta))$ implies $\phi(\eta) = \eta^{\frac{1}{n}}$, $g(\phi(\eta)) = n\eta^{\frac{n-1}{n}}$. The equilibrium conditions are

$$c'(x) = \int_0^1 \frac{f(\eta)}{g(\phi(\eta))} f(\eta)\, d\eta = \frac{1}{n}\int_0^1 \eta^{\frac{1-n}{n}}\, d\eta = 1,$$

$$c'(y) = \int_0^1 \frac{g(\varepsilon)}{f(\phi^{-1}(\varepsilon))} g(\varepsilon)\, d\varepsilon = n^2 \int_0^1 \varepsilon^{2n-2}\, d\varepsilon = \frac{n^2}{2n-1}.$$

The product of the marginal costs equals $\frac{n^2}{2n-1} > 1$ for $n > 1/2$ and $n \neq 1$. Efficiency requires that the product equals 1, which it only does for $n = 1$, that is, when $f = g$.

The example provides additional intuition for the inefficiency result. Let $n = 2$, so that the density function for men, $g(\varepsilon) = 2\varepsilon$ on [0,1]. The incentive for investment for a woman at any value of η depends upon the ratio of the densities, $\frac{f(\eta)}{g(\phi(\eta))}$. This ratio exceeds one for low values of η, but is less than one for high values of ε. Conversely, for men, the incentive to invest depends upon the inverse of this ratio, $\frac{g(\varepsilon)}{f(\phi^{-1}(\varepsilon))}$, which is low at low values of ε but high at high values of ε. In other words, the ratio of the densities plays opposite roles for the two sexes. However, the weights with which these ratios are aggregated differs between the sexes; high values of ε are given relatively large weight in the case of men, since $g(\varepsilon)$ is large in this case, while they are given relatively less weight in the case of women.

3.2 | The effects of changes in the sex ratio

One of the most concerning social phenomena of recent years is the growing inequality in the gender ratio in certain countries. This imbalance is very large in China, where it is expected that one in five boys born in 2000 will be unable to find a marriage partner (see Bhaskar, 2011). Wei and Zhang (2011) argue that the sex ratio imbalance is one of the causes of the high savings rate in China. Parents of a boy respond to the gender ratio by saving more in order to improve his chance of finding a wife, thus raising the overall savings rate. However, one might conjecture that this might be counter-balanced by the reduced saving of parents of girls. What overall economic effect is predicted by our models?

If use the deterministic model of the first part of this section, the efficiency result will still hold. So, if we only look at monotone equilibria, then they will be efficient irrespective of the gender ratio. However, the multiplicity of equilibria implies that comparative statics using only monotone equilibria may not tell the whole story. Thus, I use the noisy model to investigate the effect of an uneven sex ratio.

The principal assumption for this analysis is that the relative measure of women equals $r < 1$ or, equivalently the relative measure of men is μ where $\mu = 1/r > 1$. At the matching stage, since $r \leq 1$, all women should be matched. But under our standard assumption of assortative matching, only the top r proportion of men will be successful, with the $1 - r$ lowest ranked males failing to match.

The matching function will change too. A woman of type η will match with a man of type $\phi(\eta, r)$, where we replace Equation (20) with

$$\phi(\eta; r) = G^{-1}(1 - r(1 - F(\eta))). \tag{24}$$

A man with shock $\tilde{\varepsilon} = \phi(\underline{\eta})$, where $G(\tilde{\varepsilon}) = 1 - r$, matches with the woman with the lowest shock. The $1 - r$ men with shocks below $\tilde{\varepsilon}$ remain unmatched and I assume receive a match value of $q = 0 \leq \underline{\eta}$. The derivative of this matching function is given by

$$\phi'(\eta; r) = \frac{r f(\eta)}{g(\phi(\eta; r))}. \tag{25}$$

That is, an increase in η now increases a woman's match quality relatively more slowly, given the distribution of men is broader, since $r < 1$.

What happens if the gender ratio becomes more unbalanced? First, the number of unmatched men rises. So if $r_B < r_A$, then $\tilde{\varepsilon}_B = G^{-1}(1 - r_B) > \tilde{\varepsilon}_A = G^{-1}(1 - r_A)$. This is illustrated in Figure 4. Second, under reasonable conditions, it leads to a decrease in the slope of the matching function faced by women and thus an increase for men. We have seen in previous sections that, in the first-order condition (22), the marginal benefit to investment in terms of an improved match is determined by the slope of the matching function. So an increase (decrease) in that slope will increase (decrease) the incentive to invest.

The reason why this second effect is not trivial is that the matching function itself $\phi(\eta, r)$ depends on r. Following the research of Hoppe et al. (2009), we can identify that a sufficient condition for the intuitive response is that the distribution G has an increasing failure or hazard rate, where the hazard/failure rate is $\lambda = g(\varepsilon)/(1 - G(\varepsilon))$. While a decreasing failure/hazard rate

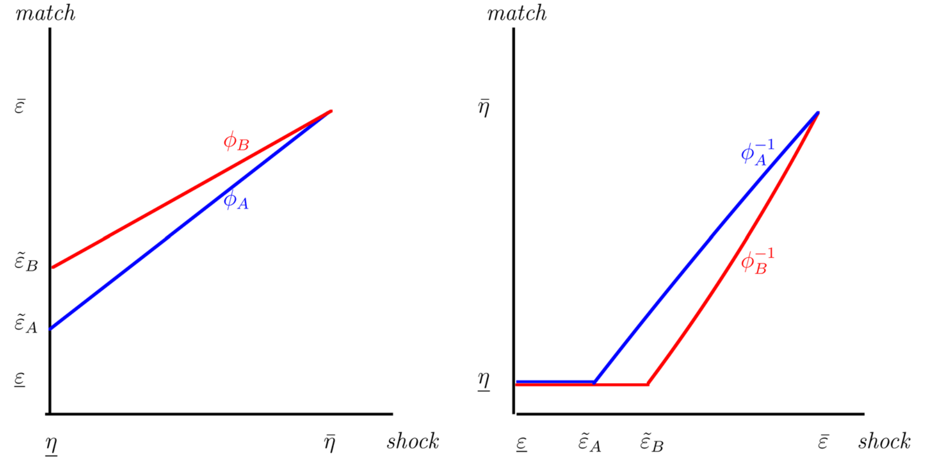

FIGURE 4 A decrease in the relative number of women such that $r_B < r_A$ leads to a decrease in slope of the matching function ϕ for women and an increase in the slope of the matching function ϕ^{-1} for men provided the distribution of men's shocks has increasing hazard rate [Colour figure can be viewed at wileyonlinelibrary.com]

is possible if the density $g(\varepsilon)$ decreases sufficiently quickly, most common distributions have an increasing failure rate.[10]

Proposition 6. *Suppose that the relative number of women decreases so that $r_B < r_A$. Assume that $G(\varepsilon)$ has an increasing hazard rate (that is, it is IFR). Then the matching function for women becomes flatter, $\phi'_B(\eta; r_B) < \phi'_A(\eta; r_A)$ (so that it is steeper for men).*

The first-order condition for women in an equilibrium where all men invest the same amount y^*, while all women invest the same amount x^* is given by

$$r \int_{\underline{\eta}}^{\bar{\eta}} \frac{f(\eta)}{g(\phi(\eta; r))} g(\eta)\, d\eta = c'(x^*). \tag{26}$$

As compared to our previous analysis, we notice two differences. The first term is the improvement in match quality, and the sparseness of women increases the investment incentives, due to the term in $1/r$.

Similarly, the first-order condition for investment for men is given by

$$\frac{1}{r} \int_{\tilde{\varepsilon}(r)}^{\bar{\varepsilon}} \frac{g(\varepsilon)}{f(\phi^{-1}(\varepsilon; r))} g(\varepsilon)\, d\varepsilon + g(\tilde{\varepsilon}(r))(\underline{\eta} + x^*) = c'(y^*). \tag{27}$$

Notice here that the gender ratio $r < 1$ increase investment incentives. Additionally, an increment in investment raises the probability of one's son getting matched, at a rate $f(\tilde{\varepsilon})$, and the marginal payoff equals the difference between matching with worst-quality girl and receiving $\underline{\eta} + x^*$, and not being matched and receiving zero.

BH16 show both the existence and uniqueness of equilibrium in this context, and show that investment is inefficient when $r < 1$. Again, we only have efficient investment when the two genders are entirely symmetric. Here, to be precise, women will invest less than the efficient amount, and men, faced with greater competition will invest more. Total investment will be greater than Pareto optimal.

Example 2. Assume that $f(\eta) = 2\eta$ and $g(\varepsilon) = 2\varepsilon$, both on $[0, 1]$ and that $c(x) = x^2/2$ and similarly for $c(y)$. Because these densities are increasing, the distributions have increasing hazard rates (IFR). It then follows from Proposition 6 and consideration of Equation (26) that investment by women is unambiguously increasing in r, so it decreases if r falls. In contrast, the matching incentive for boys increases as the sex ratio becomes more uneven. However, the overall effect on boys' investments is ambiguous, as the LHS of Equation (27) also depends on x^*, which will decrease as r falls.

However, given these specific functional forms, we can calculate explicit expressions for x^* and y^* and verify that x^* fall with r but y^* increases as the gender ratio becomes more uneven. First, $\phi(\eta; r) = \sqrt{1 - r(1 - \eta^2)}$ so that $\phi'(\eta; r) = r\eta/\sqrt{1 - r(1 - \eta^2)}$, which is increasing in r. Second, $x^* = 1 - (1 - r)\sinh^{-1}(\sqrt{\frac{r}{1-r}})/\sqrt{r}$ and $y = 1 + (1 - r)\tanh^{-1}(\sqrt{r})/\sqrt{r} + 2(\sqrt{1 - r})x^*$. Total investment, equal to $x + y/r$, does increase as r decreases below 1. However, investment is inefficiently high. As BH16 note, the efficiency condition here will be that $c'(x^*) \times c'(y^*) = r$ because there is a lower social return to men's investment, given that $1 - r$ of them will not have a partner to benefit. But one can verify that here $c'(x^*) \times c'(y^*) = x^* y^*$ is greater than 1 for $r < 1$ and therefore is much greater than the social optimum.

In summary, under some assumptions, a change in the sex ratio can indeed increase investment incentives for men, but will also decrease incentives for women. The predicted effect on total investment is consequently ambiguous. An uneven gender ratio increases the relative weight of boys in the population, and their increased investment may be enough to increase the total, but this is not guaranteed. But one can also show that an unbalanced gender ratio results in investment that overall is inefficient.

4 | FURTHER LITERATURE AND FURTHER DIRECTIONS

This paper so far has described some baseline models of matching tournaments. However, there is also an expanding literature that extends the theory and applies it in different contexts. I will now try to summarize this work. As mentioned in the introduction, there is another whole branch of literature that considers matching tournaments with TU. This is the case where the surplus from matching can be freely divided between the participants. For example, imagine a large number of workers who seek to match a large number of firms each having one job. As before, attractiveness to the other side depends on previous investments, but in addition in the TU case, each firm chooses a wage to offer to workers.[11] The first works in this direction were Cole et al. (2001a) and Felli and Roberts (2000, 2016) who both investigate efficiency in models of two-sided investment and matching. They both find that efficient investments can be sustained in equilibrium but there can be other inefficient equilibria. More recently, Nöldeke and Samuelson (2015) generalizes the earlier results, including allowing for some non-transferability, while Chade and Lindenlaub (2022) introduce stochastic returns to investment.

Returning to the NTU literature, there are many works on matching and status. As already remarked, one of the fundamental contributions of Cole et al. (1992) was the discovery that matching tournaments induce a concern with ordinal relative position—how one ranks in comparison with others. Pesendorfer (1995) uses a matching setting to investigate the incentives for consumers to buy fashion goods while at the same time suppliers of fashion goods change their designs over time. This raises the issue of signaling by wasteful expenditure analyzed in Section 2.2. Rege (2008), Hoppe et al. (2009), and Bidner (2010) investigate the trade-off between wasteful expenditure and the increase in welfare from the matching it facilitates. There has also been a use of signaling in explaining the evolution of altruism by Gintis et al. (2001) and Hopkins (2014), in the sense of providing costly public goods might be a visible signal that leads to improved matches.

Fernández and Galí (1999) is unusual in that it looks at a matching tournament model with two dimensional types, ability and wealth. Agents are students seeking to match with colleges. A tournament admission scheme is compared with a competitive market, under credit constraints and without. While the tournament is wasteful because effort is not productive, it provides more efficient matching than the market in the presence of credit constraints. Bilancini and Boncinelli (2014) make the point that with informational frictions the induced relative concerns are no longer ordinal but cardinal. That is, the distance between oneself and others now matters.

One fascinating aspect of ordinal relative concerns first investigated by Robson (1992) is that it can induce gambling in individuals that, in the absence of relative concerns, would be risk averse. Given that matching tournaments induce ordinal relative concerns, the same effect can be found there, with agents willing to take gambles before participating in matching (Cole et al., 2001b; Zhang, 2020). Becker et al. (2005) make a link between risk-taking and inequality, showing that, first, high equality will increase risk-taking and thus the inequality of wealth will increase. Thus equality is not stable. Hopkins (2018) points out that risk-taking also depends on inequality of matches or rewards, so that if this form of inequality is sufficiently low, then a very equal distribution of wealth is stable.

Cole et al.'s (1992) seminal work placed the matching tournament inside a growth model. Corneo and Jeanne (1999) extend this model to consider the effect of inequality—as we have seen, greater equality increases competition. But in a dynamic model where investment is productive, this will increase growth. However, where expenditure is wasteful, greater equality lowers growth (Hopkins and Kornienko, 2006). Thus, while matching tournaments imply a relationship between inequality and growth, the direction of the relationship depends crucially on what form investments take.

Where else could matching tournaments be applied in the future? One potential area is understanding intergenerational dynamics and mobility. That is, individuals in the current generation compete for relative position as they know that will benefit their children. That is, the current poor compete to move up, while the current rich compete to maintain their position. A recent paper in this direction is Peng (2021). Second, a different possible extension would be towards multiple matching tournaments—one could think of these as labor markets in different locations. Location choice as well as educational investments would therefore become strategic. While the existing literature in this area (see Moretti (2004), for example) allows for negative externalities in regional labor markets, the main theoretical arguments are based on signaling theory. As remarked in Section 2.2, signaling in matching markets works somewhat differently and therefore offers different possibilities. One can also consider multidimensional investments as in Bhaskar et al. (2022) where households invest both in children's human capital and in housing.

Finally, it is possible to use the theoretical approaches described here in empirical studies. For example, Bhaskar (2011) combines theoretical analysis with data from India and China. He observes that gender ratios are uneven, suggesting that parents are in effect choosing the gender of their children. This makes gender selection a form of matching tournament as costly investment in gender selection affects future marriage prospects. Bhaksar then argues, based on a theoretical model, that such sex selection is inefficient due to its creating a negative congestion externality in marriage matching. In influential work, Wei and Zhang (2011) advance the hypothesis that such an unbalanced gender ratio can affect savings decisions through the marriage market, in a similar way to the comparative statics results in Section 3.2. Specifically, households with sons in China have a particular incentive to save to improve the marriage prospects of the son. Wei and Zhang test this hypothesis and indeed find that savings by otherwise identical households with a son are higher in regions in China with a higher local sex ratio. Similar predictions were tested in a laboratory experiment by Fang et al. (2015).

More recently, Anderberg et al. (2020) and Bhaskar et al. (2022) investigate empirically how marriage matching is influenced by investments in education. The first study uses educational reform as a natural experiment to investigate how increased education improves marriage market outcomes. In the second, the authors consider how investment in children's education versus investment in their housing have different effects on attractiveness on the marriage market. The authors find that housing, because it is a good equally enjoyed by both spouses, is more attractive to potential wives than investment in a son's human capital. Thus, when the sex ratio is uneven and men face a more competitive marriage market, families invest more in housing. Overall, there is significant support for the fundamental principle of matching tournaments: future participation in a matching market influences saving and investment behavior in the present.

5 | CONCLUSIONS

This survey has set out the basics of matching tournaments, a class of models first introduced in the work of Cole et al. (1992), which maintain strategic considerations even when the number of participants is large. Matching tournaments give us a way of modeling positional externalities and relative concerns without appealing to social preferences or other behavioral assumptions. An advantage of this approach is that these concerns are endogenous, and will change when the equilibrium changes.

The models are still a long way from standard competitive markets. In general, the equilibria of matching tournaments involve inefficiently high investment. Further, the level of efficiency varies with inequality in an interesting way. In particular, there are two distinct forms of inequality, inequality in endowments and in rewards, that have opposing effects.

It is also possible for both sides of the matching market to make investments, for example both men and women invest to improve their marriage prospects. Then, when the return to investment is deterministic, investment must be efficient in any strictly monotone equilibrium. However, there are many other inefficient equilibria. Introducing noise to investment makes the equilibrium unique but generically inefficient, with investment being too high.

The hope is that this survey will open up new developments and applications for these models and methods. In particular, when there is a significant interest in inequality and its effects, I think it is important to consider, an important class of models where inequality has a direct effect on behavior, efficiency, and welfare.

ACKNOWLEDGMENTS

I thank V. Bhaskar and Tatiana Kornienko both for their collaboration on work on this area and for helpful comments on this survey. I also thank two anonymous referees for their constructive comments.

ORCID

Ed Hopkins https://orcid.org/0000-0001-9809-0312

ENDNOTES

[1] This joke is cited by Blumberg (2011), but the original author is unknown.

[2] These models are sometimes instead called contests or all-pay auctions.

[3] The proof that PAM is the unique stable matching under monotone preferences, and assuming that the two populations are equal and that one woman matches with one man, is due to Eeckhout (2000). An extension to measure-preserving matching in a continuum is given in Hopkins (2012).

[4] This technique of maximizing with respect to an agent's hypothetical type $\hat{s}$ is widely used in analyzing games of incomplete information, because it is simpler in this context. The alternative, writing utility as $T(s(x))(s-x)$ where $s(x)$ is the inverse of $x(s)$, and differentiating with respect to $x$generates the same result.

[5] Note, though, that these papers analyze models where agents, rather than seeking matches, seek status or rewards. But given that these models assume that rewards are assigned assortatively on the basis of expenditure on x, the analysis is essentially the same.

[6] This difference is similar to that between auctions, where the seller's profit depends on the buyer's action (her bid) and not her type, and pure signaling, where the receiver's final payoff depends on the sender's type not his action.

[7] This follows from the famous result of Becker (1973) that positive assortative matching is efficient when men's and women's types are complements in household production.

[8] Alternatives closer to the preferences of the previous section would also work.

[9] Attempts to give a better foundation to the matching tournament while keeping deterministic investment include Peters (2007, 2009) and Dizdar et al. (2019).

[10] In the DFR case, it is possible that an increase in the relative number of men will decrease the incentive for men to invest because the change pushes the marginal man into a thinner part of the distribution.

[11] There is also a related literature on assignment—essentially matching problems under TU but without investment decisions. Some recent works include Costrell and Loury (2004), Suen (2007), and Gola (2021).

[12] Note that $\tilde{F}(x)$ and $\tilde{F}^{-}(x)$ are only distinct when a positive mass choose the same investment $\hat{x}$. Denote $\bar{r} = \tilde{F}(\hat{x})$ and $\underline{r} = \tilde{F}^{-}(\hat{x})$, then the average value of matches ranked between $\underline{r}$ and $\bar{r}$ is $v = \int_{\underline{r}}^{\bar{r}} G^{-1}(r)\, dr/(G(\bar{r}) - G(\underline{r}))$ and by the mean value theorem there is a $\theta \in (0,1)$ such that $G^{-1}(\theta\tilde{F}(x) + (1-\theta)\tilde{F}^{-}(x)) = v$.

[13] This is shown, for example, for the concave order in Shaked and Shanthikumar (2007, Ch3), which is equivalent to SOSD with equal means.

[14] This is equivalent to F being a DFR (decreasing failure rate) distribution due to the result of Barlow and Proschan (1965).

REFERENCES

Anderberg, D., Bagger, J., Bhaskar, V., & Wilson, T. (2020). *Marriage market equilibrium with matching on latent ability: Identification using a compulsory schooling expansion* [Working paper].

Barlow, R. E., & Proschan, F. (1965). *Mathematical theory of reliability*. Wiley.

Becker, G. S. (1973). A theory of marriage: Part 1. *Journal of Political Economy*, *81*, 813–846.

Becker, G. S., Murphy, K. M., & Werning, I. (2005). The equilibrium distribution of income and the market for status. *Journal of Political Economy*, *113*(2), 282–310.

Bhaskar, V. (2011). Sex selection and gender balance. *American Economic Journal: Microeconomics*, *3*(1), 214–44.

Bhaskar, V., & Hopkins, E. (2016). Marriage as a rat race: Noisy pre-marital investments with assortative matching. *Journal of Political Economy*, *124*(4), 992–1045.

Bhaskar, V., Li, W., & Yi, J. (2022). *Multidimensional pre-marital investments with imperfect commitment* [Working paper].

Bidner, C. (2010). Pre-match investment with frictions. *Games and Economic Behavior*, *68*(1), 23–34.

Bilancini, E., & Boncinelli, L. (2014). Instrumental cardinal concerns for social status in two-sided matching with non-transferable utility. *European Economic Review*, *67*, 174–189.

Blumberg, M. (2011). Outrunning the bear, Business Insider. Retrieved from https://www.businessinsider.com/outrunning-the-bear-2011-10

Chade, H., & Lindenlaub, I. (2022). Risky matching. *Review of Economic Studies*, *89*(2), 626–665.

Cole, H. L., Mailath, G. J., & Postlewaite, A. (1992). Social norms, savings behavior, and growth. *Journal of Political Economy*, *100*(6), 1092–1125.

Cole, H. L., Mailath, G. J., & Postlewaite, A. (1995). Incorporating concern for relative wealth into economic models. *Federal Reserve Bank of Minneapolis Quarterly Review*, *19*(3), 12–21.

Cole, H. L., Mailath, G. J., & Postlewaite, A. (1998). Class systems and the enforcement of social norms. *Journal of Public Economics*, *70*, 5–35.

Cole, H. L., Mailath, G. J., & Postlewaite, A. (2001a). Efficient non-contractible investments in large economies. *Journal of Economic Theory*, *101*, 333–373.

Cole, H. L., Mailath, G. J., & Postlewaite, A. (2001b). Investment and concern for relative position. *Review of Economic Design*, *6*, 241–261.

Coles, P., Kushnir, A., & Niederle, M. (2013). Preference signaling in matching markets. *American Economic Journal: Microeconomics*, *5.2*, 99–134.

Corneo, G., & Jeanne, O. (1999). Social organization in an endogenous growth model. *International Economic Review*, *40*(3), 711–726.

Costrell, R. M., & Loury, G. C. (2004). Distribution of ability and earnings in a hierarchical job assignment model. *Journal of Political Economy*, *112*, 1322–1363.

Dizdar, D., Moldovanu, B., & Szech, N. (2019). The feedback effect in two-sided markets with bilateral investments. *Journal of Economic Theory*, *182*, 106–142.

Eeckhout, J. (2000). On the uniqueness of stable marriage matchings. *Economics Letters*, *69*, 1–8.

Fang, H., Gong, B., & Wei, S. J. (2015). *An experimental study of the competitive saving motive* [Working paper].

Felli, L., & Roberts, K. W. S. (2000). *Does competition solve the hold-up problem?* [CARESS Working paper #00-04].

Felli, L., & Roberts, K. (2016). Does competition solve the hold-up problem?*Economica*, *83*(329), 172–200.

Fernández, R., & Galí, J. (1999). To each according to...? Markets, tournaments and the matching problem with borrowing constraints. *Review of Economic Studies*, *66*, 799–824.

Frank, R. H. (1985). The demand for unobservable and other positional goods. *American Economic Review*, *75*(1), 101–116.

Frank, R. H. (1991). Positional externalities. In R.Zeckhauser (Ed.), *Strategy and choice: Essays in honor of Thomas C. Schelling* (pp. 2547). MIT Press.

Gale, D., & Shapley, L. S. (1962). College admissions and the stability of marriage. *The American Mathematical Monthly*, *69*(1), 9–15.

Gintis, H., Smith, E. A., & Bowles, S. (2001). Costly signaling and cooperation. *Journal of Theoretical Biology*, *213*, 103–119.

Gola, P. (2021). Supply and demand in a two-sector matching model. *Journal of Political Economy*, *129*(3), 940–978.

Hopkins, E. (2012). Job market signalling of relative position or Becker married to spence. *Journal of the European Economic Association*, *10*(2), 290–322.

Hopkins, E. (2014). Competitive altruism, mentalizing, and signaling. *American Economic Journal: Microeconomics*, *6*(4), 272–292.

Hopkins, E. (2018). Inequality and risk-taking behaviour. *Games and Economic Behavior*, *107*, 316–328.

Hopkins, E., & Kornienko, T. (2004). Running to keep in the same place: Consumer choice as a game of status. *American Economic Review*, *94*(4), 1085–1107.

Hopkins, E., & Kornienko, T. (2006). Inequality and growth in the presence of competition for status. *Economics Letters*, *93*, 291–296.

Hopkins, E., & Kornienko, T. (2009). Status, affluence, and inequality: rank-based comparisons in games of status. *Games and Economic Behavior*, *67*, 552–568.

Hopkins, E., & Kornienko, T. (2010). Which inequality? The inequality of endowments versus the inequality of rewards. *American Economic Journal: Microeconomics*, *2*(3), 106–137.

Hoppe, H. C., Moldavanu, B., & Sela, A. (2009). The theory of assortative matching based on costly signals. *Review of Economic Studies*, *76*, 253–281.

Konrad, K. (2009). *Strategy in contests – an introduction*. Oxford University Press.

Mailath, G. J. (1987). Incentive compatibility in signaling games with a continuum of types. *Econometrica*, *55*(6), 1349–1365.
Moretti, E. (2004). Human capital externalities in cities. In *Handbook of regional and urban economics* (Vol. 4, pp. 2243–2291). Elsevier.
Nöldeke, G., & Samuelson, L. (2015). Investment and competitive matching. *Econometrica*, *83*(3), 835–896.
Peng, B. (2021). Positional competition: A theory of the Great Gatsby curve and the Easterlin paradox. *Journal of Economic Behavior and Organization*, *186*, 562–575.
Pesendorfer, W. (1995). Design innovation and fashion cycles. *American Economic Review*, *85*(4), 771–792.
Peters, M. (2007). The pre-marital investment game. *Journal of Economic Theory*, *137*, 186–213.
Peters, M. (2009). *Truncated hedonic equilibrium* [Working paper].
Peters, M., & Siow, A. (2002). Competing premarital investments. *Journal of Political Economy*, *110*, 592–608.
Rege, M. (2008). Why do people care about social status?*Journal of Economic Behavior and Organization*, *66*, 233–242.
Robson, A. (1992). Status, the distribution of wealth, private and social attitudes to risk. *Econometrica*, *60*(4), 837–887.
Roth, A. E. (1984). The evolution of the labor market for medical interns and residents: A case study in game theory. *Journal of Political Economy*, *92*, 991–1016.
Shaked, M., & Shanthikumar, J. G. (2007). *Stochastic orders*. Springer.
Suen, W. (2007). The comparative statics of differential rents in two-sided matching markets. *Journal of Economic Inequality*, *5*, 149–158.
Wei, S., & Zhang, X. (2011). The competitive saving motive: Evidence from rising sex ratios and savings rates in China. *Journal of Political Economy*, *119*, 511–564.
Zhang, H. (2020). Pre-matching gambles. *Games and Economic Behavior*, *121*, 76–89.

How to cite this article: Hopkins, E. (2022). Is everything relative? A survey of the theory of matching tournaments. *Journal of Economic Surveys*, 1–27.
https://doi.org/10.1111/joes.12508

APPENDIX: PROOFS

Proof of Proposition 1. This is a sketch based on the longer proof in Hopkins and Kornienko (2004). The first step is to show, using results from auction theory, that best responses are increasing in type s. Then one can show that a symmetric equilibrium strategy $x(s)$ must be strictly increasing in s. If not, there would be a mass of agents choosing the same investment x. However, Hopkins and Kornienko (2004) make the further assumption that if there is a tie in investment, matches are assigned by the rule

$$T(x, \tilde{F}(\cdot)) = G^{-1}(\theta\tilde{F}(x) + (1-\theta)\tilde{F}^{-}(x)) \tag{A1}$$

where $\tilde{F}^{-}(x) = \lim_{\xi\uparrow x}\tilde{F}(\xi)$ and for some $\theta \in (0,1)$. For example, this rule would be consistent with the corresponding range of matches being uniformly randomly assigned to those households.[12] Then, individuals always have an incentive to increase investment to break the tie and therefore the equilibrium strategy must be strictly increasing. This further implies that the equilibrium strategy is both continuous and differentiable. Given that investment is continuous and strictly increasing, the only stable matching is positive assortative and takes the form given by the matching function $T(s)$as defined in Equation (3), giving rise to the reduced form payoffs $U(x(\hat{s}), s, T(\hat{s}))$. Thus, given this and differentiability of $x(s)$, the equilibrium strategy satisfies the first-order condition (4), which gives rise to the ODE (5) that has a unique solution by the fundamental theorem of differential equations. The efficiency result is outlined in the main text. □

Proof of Proposition 2. First, note that, for two distributions, F_A, F_B, with the same means $F_A \succ_{SOSD} F_B$ if and only if

$$\int_0^s F_A(u)\,du \le \int_0^s F_B(u)\,du \Leftrightarrow \int_0^r F_A^{-1}(u)\,du \ge \int_0^r F_B^{-1}(u)\,du, \tag{A2}$$

for all $s \in [\underline{s}, \bar{s}]$ and for all $r \in [0,1]$ respectively.[13] Under the assumption in (a) that G is uniform then from Equation (7), one has $U(s) = \int_0^s F(u)\,du$ and the result on utility follows. Further, note that from Equation (6), when G is uniform, $x(s) = E[S|S < s]$. Then from the results of Shaked and Shanthikumar (2007, p118), it follows that $x_A(s) \ge x_B(s)$ for all s. Under the assumption in (b) that F is uniform so that $r = F(s) = s$, $U(s) = \int_0^s G^{-1}(u)\,du$ and again the result on utility follows from Equation (A2). Finally, when F is uniform, we have $x(s) = \int_0^{G^{-1}(s)} G(t)\,dt/G^{-1}(s) = E[T|T < G^{-1}(s)]$ and the result follows. □

Proof of Proposition 3. If $F = G$, then $T(s) = s$ and $U(s) = s^2/2$ from Equation (7). Total welfare across the female population is $\int_0^1 s^2/2\,dF(s) = (Var[s] + E[s]^2)/2$. The payoff to random matching, given zero signaling, is $E[t]s$ and the total welfare is $E[s]^2$, because if $F = G, E[s] = E[t]$. Thus, assortative matching is better if $Var[s] > E[s]^2$, equivalently if $CV[s] = \sqrt{Var[s]}/E[s] > 1$.[14] □

Proof of Proposition 5. Existence and uniqueness follow from Theorem 1 of Bhaskar and Hopkins (2016). To address efficiency, it is useful to make the following change in variables in the first-order condition for men. Since $\varepsilon = \phi(\eta)$,

$$d\varepsilon = \phi'(\eta)\,d\eta = \frac{f(\eta)}{g(\phi(\eta))}\,d\eta.$$

Thus the first-order condition for men is rewritten as

$$\int_{\underline{\eta}}^{\bar{\eta}} g(\phi(\eta))\,d\eta = c'(y).$$

Consider the product of the two first-order conditions:

$$c'(x) \times c'(y) = \left(\int_{\underline{\eta}}^{\bar{\eta}} \frac{f(\eta)}{g(\phi(\eta))} f(\eta)\,d\eta\right)\left(\int_{\underline{\eta}}^{\bar{\eta}} g(\phi(\eta))\,d\eta\right).$$

By the Cauchy–Schwarz inequality

$$\left(\int_{\underline{\eta}}^{\bar{\eta}} \frac{f(\eta)}{g(\phi(\eta))} f(\eta)\,d\eta\right)\left(\int_{\underline{\eta}}^{\bar{\eta}} g(\phi(\eta))\,d\eta\right) \ge \left[\int_{\underline{\eta}}^{\bar{\eta}} \left(\frac{f(\eta)}{(g(\phi(\eta)))^{1/2}}\right)(g(\phi(\eta)))^{1/2}\,d\eta\right]^2 = 1,$$

with the inequality being strict if the two terms are linearly independent. Thus $c'(x) \times c'(y) > 1$ if $\frac{f(\eta)}{\sqrt{g(\phi(\eta))}}$ and $\sqrt{g(\phi(\eta))}$ are linearly independent functions of η. Since Pareto efficiency requires $c'(x) \times c'(y) = 1$, we have overinvestment generically if the distributions f and g differ. □

Proof of Proposition 6. First, note that as r falls, the slope $\phi'(\tilde{\varepsilon}; r)$ of the matching function will fall. We have $\phi'(\eta; r) = rf(\eta)/g(\phi(\eta; r))$. Differentiating, and substituting r with $1 - G(\phi(\underline{\eta}))$, ϕ' is increasing in r if

$$g^2(\phi(\eta)) + rg'(\phi(\eta)) = g^2(\phi(\eta)) + g'(\phi(\eta))(1 - G(\phi(\underline{\eta}))) > 0.$$

It is easily verified that this inequality holds at $\tilde{\varepsilon} = \phi(\underline{\eta})$ if and only if G has an increasing hazard rate at $\tilde{\varepsilon}$. But since $1 - G(\varepsilon)$ has its maximum value on $[\tilde{\varepsilon}, \bar{\varepsilon}]$ at $\tilde{\varepsilon}$, the inequality holds for all $\varepsilon \in [\tilde{\varepsilon}, \bar{\varepsilon}]$ and thus for all $\eta \in [\underline{\eta}, \bar{\eta}]$. □