

## A Note on Best Response Dynamics

Ed Hopkins<sup>\*,†</sup>

*Department of Economics, University of Edinburgh, Edinburgh EH8 9JY,  
United Kingdom*

Received March 25, 1997

We investigate the relationship between the continuous time best response dynamic, its perturbed version, and evolutionary dynamics in relation to mixed strategy equilibria. We find that as the level of noise approaches zero, the perturbed best response dynamic has the same qualitative properties as a broad class of evolutionary dynamics. That is, stability properties of equilibria are robust across learning dynamics of quite different origins and motivations. *Journal of Economic Literature* Classification Numbers: C72, D83. © 1999 Academic Press

*Key Words:* games; learning; evolution; mixed strategies.

### 1. INTRODUCTION

This article addresses the proliferation of different learning models to be found in the recent literature and illuminates some common characteristics. There is some debate about the applicability of evolutionary dynamics arising from biological models within the social sciences. They are often contrasted with belief-based models of learning such as fictitious play which, though assuming some bounded rationality, do involve some optimization. This article compares three types of dynamics. These are the continuous time best response (*BR*) dynamic, the smoothed best response ( $\overline{BR}$ ) dynamic, and a general class of evolutionary dynamics called adaptive dynamics (Hofbauer and Sigmund, 1990) or positive definite dynamics (Hopkins, 1995). Here we show that, whatever other differences they may possess, these learning dynamics have, for the most part, the same asymptotic properties.

Underlying most of the recent work on learning in games has been the model of fictitious play first introduced in the early days of game theory.

\*I thank Josef Hofbauer and Ken Binmore for helpful comments and suggestions and Drew Fudenberg for asking me the question that produced this article.

<sup>†</sup>E-mail address: E.Hopkins@ed.ac.uk.



Players construct an estimate of their opponents' strategies from past play and then choose a strategy which is a best response to this estimate. But fictitious play can be difficult to analyze because there can be abrupt jumps in play. The dynamics can be smoothed by various methods. The first is to look at dynamics in continuous time as do Gilboa and Matsui (1991) and Hofbauer (1995). These we call the *BR* dynamics. Second, it is possible to go further and perturb payoffs. With the addition of suitable noise, the best response correspondence becomes a smooth best response function and the transformation to a smooth dynamical system, the *BR* dynamics, is complete.

Evolutionary dynamics obviously have different origins, arising in the study of animal behavior. However, it was found more recently that the evolutionary replicator dynamics can arise from certain models of human learning (Börgers and Sarin, 1995; Schlag, 1998). Furthermore, in an earlier article (Hopkins, 1995) it was shown that, if either fictitious play or another type of learning dynamic often considered in the literature and known as stimulus–response or reinforcement learning, were aggregated over a large population of players, the resulting dynamic possessed the same basic qualitative properties as the evolutionary replicator dynamics. More precisely, they all can be represented as smooth symmetric positive definite transformations of payoffs. We can therefore consider a class of dynamics satisfying this simple property and we name them positive definite adaptive (*PDA*) dynamics.<sup>1</sup>

Thus, these earlier results showed an equivalence of fictitious play and evolutionary dynamics at the level of a population of players. It is natural to wonder whether evolutionary dynamics have any relevance to learning at the level of the individual and whether they can help to explain learning behavior apparent in, for example, experimental data. The link is the theory of stochastic approximation which allows analysis of discrete time stochastic processes by looking at deterministic continuous time dynamics. Recent research showed how there is a relationship between *BR* dynamics and stochastic fictitious play (Benaïm and Hirsch, 1999). The question is then the relationship, if any, between best response and evolutionary dynamics. Whatever it is, it is not straightforward. For example, in the class of  $2 \times 2$  games which possess a unique mixed strategy equilibrium, *BR* and *BR* dynamics converge whereas *PDA* dynamics may not. Yet otherwise the stability properties of *BR* dynamics as outlined in Hofbauer (1995) are almost identical to those of *PDA* dynamics.

<sup>1</sup>Hofbauer and Sigmund's usage (1990) of "adaptive" dynamic predates my choice of "positive definite" (Hopkins, 1995). However, the subsequent growth of the literature on learning produced completely different dynamics which were also called adaptive. *PDA* is an attempt to avoid ambiguity.

Here we are able to show the exact nature of the relationship. We first consider the perturbed best response  $\overline{BR}$  dynamics. We see that as the level of noise approaches zero, the  $\overline{BR}$  dynamics approach both the  $BR$  and  $PDA$  dynamics. For low levels of noise all three dynamics possess the same stability properties for many games.

## 2. BEST RESPONSE LEARNING DYNAMICS

We consider learning in the context of two-player normal-form games, formally  $G = (\{1, 2\}, I, J, \pi_1, \pi_2)$ . The games may be either symmetric or asymmetric (in the evolutionary sense). In the second case, the players labelled 1 are drawn from a different “population” from the players labelled 2. For example, in the “Battle of the Sexes” game, players are matched so that a female always plays against a male.  $I$  is a set of  $n$  strategies, available to the first population,  $J$  is the set of  $m$  strategies of the second population. A symmetric game means that there is a single population of players all facing the same decision problem, i.e.,  $I = J$ , and  $\pi_1 = \pi_2$ .

In this article, we concentrate on mixed strategy equilibria. This is because, first, strict pure Nash equilibria are dynamically stable under nearly all formulations of learning dynamics. This is not therefore a point of interest. In contrast, for pure equilibria which are not strict, the results are very sensitive to formulation of the dynamics. The attraction of mixed strategy equilibria is that while in much of the literature they are seen as similarly problematic (Jordan, 1993; Fudenberg and Kreps, 1993) they actually allow a far more unified treatment than at first seems possible.

One way to interpret the  $BR$  dynamics is that within a large population a small proportion of agents adjust their strategy at any given time, changing to a strategy that is a best response to the current strategy of their opponents. If payoffs are subject to some sort of noise then it is potentially plausible that agents when called to adjust their strategy would not always choose a best response. This is one way to derive the  $\overline{BR}$  dynamics.

However, the use of best response dynamics is not confined to large population, social learning models. It turns out that standard two-player discrete time fictitious play can be approximated asymptotically by the (two-population)  $BR$  dynamic. Moreover, results from the theory of stochastic approximation show that the asymptotic behavior of  $\overline{BR}$  dynamics and stochastic fictitious play, that is, fictitious play where payoffs and/or agents’ choices are randomly perturbed, are similarly linked. The exact connections are set out in Fudenberg and Levine (1996, Chapters 2 and 4). Thus, although we confine our attention to deterministic continuous time dynamics,

our results have uses in other contexts where each population may consist of a single player.

The state of the system can be summarized by a vector  $x = (x_1, \dots, x_n)$  in the symmetric case, and in the asymmetric case by two vectors  $x$  and  $y = (y_1, \dots, y_m)$ . Given the discussion above these can ambiguously refer to the proportions of a large population pursuing each strategy or to the current strategy, possibly mixed, of an individual player. In any case,  $x \in S_n$ ,  $y \in S_m$  where  $S_n$  is the simplex  $\{x = (x_1, \dots, x_n) \in \mathbb{R}^n: \sum x_i = 1, x_i \geq 0, \text{ for } i = 1, \dots, n\}$ . If we assume payoffs are linear, an agent's expected payoff from each of her  $n$  strategies will be given by the vector  $\pi_1 = Ax$  for symmetric games and in the asymmetric case, payoffs in the first population will be  $\pi_1 = Ay$ , and payoffs in the second population will be  $\pi_2 = Bx$ . However, as noted above, we concentrate on mixed strategy equilibria and so we confine our attention to games which possess a mixed strategy equilibrium. Any such equilibrium point, we denote by  $q$ .

We have some additional problems caused by the fact that the simplex  $S_n$  is only of dimension  $n - 1$ . If  $x \in S_n$  and  $\dot{x} = f(x)$  then for  $x(t)$  the solution of the differential equation to remain in  $S_n$  it must be that  $f(x)$  is in the set  $\mathbb{R}_0^n = \{x \in \mathbb{R}^n: \sum x_i = 0\}$ . When we look at a linearization around an equilibrium of such a dynamical system, the stability of that equilibrium will be determined by  $n - 1$  eigenvalues which refer to  $\mathbb{R}_0^n$ , which has dimension  $n - 1$ , and not the  $n$ th eigenvalue which refers to the rest of  $\mathbb{R}^n$ . We therefore need to introduce the idea of a  $n \times n$  matrix  $A$  constrained to  $\mathbb{R}_0^n$ , denoting this constrained matrix  $A_0$ . Define  $A_0 = C^T AC$ , where  $C$  is a  $n \times n - 1$  matrix, with the initial  $n - 1 \times n - 1$  block being the identity matrix of order  $n - 1$  with every entry in the  $n$ th row being  $-1$ . Then  $A$  is positive definite with respect to  $\mathbb{R}_0^n$ , that is,  $z \cdot Az > 0, \forall z \in \mathbb{R}_0^n \setminus \{0\}$ , iff  $x \cdot A_0 x > 0, \forall x \in \mathbb{R}^{n-1} \setminus \{0\}$ . Let  $x_{(n-1)}$  be the vector consisting of the first  $n - 1$  elements of the vector  $x$ . Then  $d(Ax)/dx_{(n-1)} = AC$  for any  $x \in S_n$ .

The  $BR$  dynamics in the symmetric case are simply specified as

$$\dot{x} = BR(x) - x, \quad (1)$$

where  $BR(x)$  is the set of all best responses to  $x$ . Of course,  $BR(x)$  is therefore not a function but a correspondence and so (1) does not represent a standard dynamical system. It is still possible to subject it to detailed analysis as Hofbauer (1995) shows. However, with small changes to our specification, we can obtain the smooth  $\overline{BR}$  dynamics. Imagine an agent in a single large random-matching population playing some strategy  $z \in S_n$ . If the agent's payoffs are linear in the population state  $x$  her expected payoffs would be  $z \cdot Ax$ . Following Fudenberg and Levine (1996, Chapter 4), we suppose payoffs are perturbed such that payoffs are in fact given by

$$z \cdot Ax + \gamma v(z). \quad (2)$$

The function  $v(z): S_n \rightarrow \mathbb{R}$  has the following properties:

1.  $v(z)$  is concave on  $S_n$ . More precisely,  $v_{zz}$ , the matrix of second derivatives of  $v$  with respect to  $z_{(n-1)}$ , is negative definite.
2.  $\lim_{z_i \rightarrow 0} v_{z_i} = -\infty$  for all  $z_i$ .

Because these conditions are in fact important in the results that follow, it is worth stating their purpose. The object in perturbing payoffs is ensure that for each state  $x$ , there exists a unique best response, so that we can replace the best reply correspondence  $BR$  with the best reply function  $\overline{BR}$ . Brief reflection reveals that any other specification of the noise function  $v$  would not guarantee that (2) actually had a unique maximum on  $S_n$  and thus for every  $x$ , there would be a unique best reply. Differentiating with respect to  $z_{(n-1)}$ , the  $n - 1$  first order conditions for a maximum will be

$$v_z(z) = -\frac{C^T Ax}{\gamma}. \quad (3)$$

The second condition states that as  $z$  approaches the boundary of the simplex the slope of  $v(z)$  becomes infinite and thus ensures that the system of equations (3) possess a solution on the interior of  $S_n$ . This we write as

$$z = \overline{BR}(x) = v_z^{-1}\left(-\frac{C^T Ax}{\gamma}\right). \quad (4)$$

Now again if we consider that within this large population, there is slow and gradual adjustment toward the best response, there will be  $\overline{BR}$  dynamics of the form

$$\dot{x} = \overline{BR}(x) - x, \quad (5)$$

where  $\overline{BR}$  is the smoothed best response function as derived above. Let  $q$  be a fully mixed Nash equilibrium and thus a fixed point for the  $BR$  dynamics (1). We note that from Harsanyi's (1973) purification theorem, when  $\gamma$ , the level of perturbation, is small, there is an associated equilibrium (except in pathological cases of measure zero) of the  $\overline{BR}$  dynamics (5), which we label  $q'$  and that  $\lim_{\gamma \rightarrow 0} q' = q$ .

### 3. POSITIVE DEFINITE ADAPTIVE DYNAMICS

The version of evolutionary dynamics that we use is entitled positive definite adaptive (*PDA*) dynamics. Given that this formulation of dynamics is not well known, before defining them formally, we give a few words of justification of their use. First, they generalize the well-known replicator dynamics. The reader is welcome mentally to substitute "replicator" for each

occurrence of *PDA* if that is more familiar. The advantage of using a general class of dynamics like *PDA* is that other dynamics beside the replicator are included. The next question is then, why this generalization rather than another? Weibull's (1995) book, for example, gives many examples of differently formulated classes of dynamics. Our reason for focusing on *PDA* dynamics here is simply that they, unlike other specifications, arise directly from analysis of the *BR* dynamics. We will also go on to discuss the connection between our results and some other results on monotonic dynamics.

DEFINITION. A *PDA* dynamic is a dynamic of the form,

$$\dot{x} = Q(x)Ax, \quad (6)$$

where  $Q(x)$  is a *PDA* dynamic operator, i.e., a matrix function that on the interior of  $S_n$  possesses the following properties:

1. Positive definiteness with respect to  $\mathbb{R}_0^n$ , i.e.,  $z \cdot Qz > 0$  for all nonzero  $z \in \mathbb{R}_0^n$ .
2. Symmetry.
3.  $Q$  maps  $\mathbb{R}^n \rightarrow \mathbb{R}_0^n$ .
4. Continuous differentiability with respect to  $x$ .

Positive definiteness ensures that the angle between the vector of payoffs  $Ax$  and the vector of the dynamic  $QAx$  must be less than  $90^\circ$ .<sup>2</sup> It is thus a very weak formulation of the idea that the growth rate of strategies should be increasing in their payoffs. Condition 2 strengthens this somewhat and is necessary to establish Lemma 1 below. Condition 3 is to ensure the dynamic remains on the simplex, and condition 4 so that (6) possesses a solution and so that we can construct a linearization at an equilibrium point.

The advantage of this approach, as set out in Hofbauer and Sigmund (1990), Hopkins (1995), and Hopkins and Seymour (1996), is that it is very easy to determine the stability of any given equilibrium for the whole class of dynamic.<sup>3</sup> For example, the Jacobian of the dynamic at any fully mixed equilibrium for a linear symmetric game is given simply by  $QA$ . It is easy to show the following:

LEMMA 1. *If  $Q$  is a symmetric positive definite matrix and if  $A$  is negative (positive) definite, all the eigenvalues of  $QA$  have negative (positive) real parts.*

<sup>2</sup>Dynamics satisfying this property are called "weakly compatible" by Friedman (1991) and are called "myopic adjustment dynamics" by Swinkels (1993).

<sup>3</sup>Compare Fudenberg and Kreps' (1993, p. 340) discussion of Rock-Scissors-Paper-type games. When not zero sum, it is not a trivial matter to determine stability of equilibrium under fictitious play, whereas it is relatively easy under *PDA* dynamics. See the discussion around (10) below.

*Proof.* See, for example, Hofbauer and Sigmund (1988, p. 129). ■

This has the consequence that if a game matrix  $A$  is negative definite with respect to  $\mathbb{R}_0^n$  then the linearization  $QA$  at a fully mixed equilibrium (if it exists) will have only negative eigenvalues with respect to  $\mathbb{R}_0^n$ . Hence this equilibrium will be asymptotically stable for any  $PDA$  dynamic. Now the definition of an evolutionarily stable strategy or ESS demands that either an equilibrium  $q$  be a strict Nash equilibrium or that

$$q \cdot Ax > x \cdot Ax \quad \forall x \neq q.$$

The equation expresses the idea that a strategy  $q$  could repel an invading strategy  $x$ , if it does better against  $x$  than  $x$  does against itself. If  $q$  is a fully mixed equilibrium, then  $x \cdot Aq = q \cdot Aq$  and so we can write the above condition as  $x \cdot Ax - q \cdot Ax = (x - q) \cdot A(x - q) < 0$ . That is, the condition that  $q$  be an ESS is exactly that  $A$  should be negative definite. But there are many cases in which the game matrix is positive definite and therefore any fully mixed equilibrium of that game will be unstable under  $PDA$  dynamics. The intuition is that positive definiteness of the payoff matrix  $A$  means that the game possesses positive externalities of the type found in coordination games. Start at the mixed equilibrium where all strategies generate the same payoff. Now if  $x_i$ , the current weight placed on strategy  $i$ , increases, the positive externality means that the return to that strategy  $(Ax)_i$  also increases. Under any  $PDA$  dynamic this leads to a further increase in  $x_i$  which is clearly destabilizing. We now show that best response dynamics behave in the same way.

#### 4. RESULTS

The  $BR$  and  $\overline{BR}$  dynamics do not fit the definition of  $PDA$  dynamics as set out in the previous section. The most obvious deficiency, apart from the fact that  $BR$  dynamics are not even a function, is that they are nonlinear and  $PDA$  dynamics are linear in payoffs (even if nonlinear in  $x$ ). Nevertheless, the structure of  $PDA$  dynamics have a surprising relevance to the  $\overline{BR}$  dynamics as the following result indicates.

LEMMA 2. We can write  $d\overline{BR}(x)/dx$  as  $(1/\gamma)Q_B A$  where  $Q_B$  is a  $PDA$  dynamic operator at  $q'$ .

*Proof.* First of all note that if we write  $u = -C^T Ax/\gamma$  then from (4) we have

$$\frac{d\overline{BR}}{dx} = \frac{dz}{du} \frac{du}{dx} = \frac{1}{\gamma} Q_B A,$$

where  $Q_B = -(d\overline{BR}/du)C^T$ . If we consider  $Q_B$  constrained to  $\mathbb{R}_0^n$ , we have

$$\frac{dz_{(n-1)}}{dx_{(n-1)}} = \frac{dz_{(n-1)}}{du} \frac{du}{dx_{(n-1)}} = \left( \frac{du}{dz_{(n-1)}} \right)^{-1} \frac{du}{dx_{(n-1)}} = -\frac{1}{\gamma} v_{zz}^{-1} A_0. \quad (7)$$

By hypothesis,  $v_{zz}$  is symmetric negative definite. It follows that  $-v_{zz}^{-1}$  is symmetric positive definite. We need to show that  $Q_B: \mathbb{R}^n \rightarrow \mathbb{R}_0^n$ . Looking at (4) we can see that the perturbed best response function  $\overline{BR}$  maps payoffs  $Ax$  to a strategy  $z$ , and so we have  $\overline{BR}(x): \mathbb{R}^n \rightarrow S_n$ . Thus when a linear approximation,  $Q_B$ , is constructed, it will map  $\mathbb{R}^n$  into the set  $\mathbb{R}_0^n$ , parallel to  $S_n$  but passing through the origin. ■

We can look at the explicit functional form  $v(z) = \sum -z_i \log z_i$  which gives us what we can call the exponential best reply function,

$$\overline{BR}_i^e(x) = \frac{\exp[(1/\gamma)(Ax)_i]}{\sum_{j=1}^n \exp[(1/\gamma)(Ax)_j]}. \quad (8)$$

If this is the case then the Jacobian taken in the special case when there is an interior mixed equilibrium  $q' = q = (1/n, 1/n, \dots, 1/n)$  can be constructed from the fact that

$$\left. \frac{d\overline{BR}_i^e}{dx_j} \right|_{x=q'} = \frac{1}{\gamma} \left( a_{ij} - \frac{1}{n} \sum_{k=1}^n a_{ik} \right).$$

That is, strangely,

$$\left. \frac{d\overline{BR}^e}{dx} \right|_{x=q'} = \frac{1}{\gamma} J,$$

where  $J$  is both the Jacobian of Friedman's (1991) "linear" dynamics,

$$\dot{x}_i = (Ax)_i - \frac{1}{n} \sum_{k=1}^n (Ax)_k,$$

and also the Jacobian of the replicator dynamics (both examples of *PDA* dynamics),

$$\dot{x}_i = x_i [(Ax)_i - x \cdot Ax]$$

when evaluated at  $q = (1/n, 1/n, \dots, 1/n)$ . Thus the Jacobian of the  $\overline{BR}^e$  dynamics is given by

$$J_{\overline{BR}^e} = \frac{1}{\gamma} J - I,$$



where  $I$  is the identity matrix. Or more generally the Jacobian of any  $\overline{BR}$  dynamic at any mixed equilibrium point  $q'$ , irrespective of whether  $q' = q$ , from (5) and Lemma 2 is given by

$$J_{\overline{BR}} = \frac{1}{\gamma} Q_B A - I. \quad (9)$$

If we consider the behavior of the dynamics when the perturbation becomes very small, that is when  $\gamma \rightarrow 0$ , then the following is clear—we approach the  $PDA$  dynamics.

**PROPOSITION 1.** *Consider a fully mixed strategy equilibrium  $q$  derived from a game matrix  $A$ . If  $A$  is negative or positive definite with respect to  $\mathbb{R}_0^n$ , there exists a  $\gamma$  sufficiently small such that a perturbed equilibrium  $q'$  has the same stability properties under  $\overline{BR}$  dynamics as  $q$  has under  $PDA$  dynamics. If  $A$  is constant sum then  $q$  is a center for the linearized  $PDA$  dynamics, and  $q'$  is asymptotically stable for  $\overline{BR}$  dynamics for any  $\gamma > 0$ .*

*Proof.* The linearization of any  $PDA$  dynamic of form (6) at a fully mixed equilibrium is simply given by  $QA$ . Given that  $A$  is negative (positive) definite with respect to  $\mathbb{R}_0^n$  then from Lemma 1 all  $n - 1$  eigenvalues of  $QA$  when constrained to  $\mathbb{R}_0^n$  are negative (positive). It follows from Lemma 2 that similarly the relevant eigenvalues of  $Q_B A$  evaluated at  $q'$  must all be negative (positive). If  $\mu$  is such an eigenvalue of  $Q_B A$  then there is a corresponding eigenvalue of  $J_{\overline{BR}}$  equal to  $\mu/\gamma - 1$ . Clearly for some  $\gamma$  sufficiently close to 0 the eigenvalues of  $QA$  and  $J_{\overline{BR}}$  have the same sign pattern.

In the case of a constant sum game,  $z \cdot Az = 0, \forall z \in \mathbb{R}_0^n$ . This has the consequence that the eigenvalues of  $QA$ , if  $Q$  is a  $PDA$  operator, constrained to  $\mathbb{R}_0^n$  have zero real part. Hence the eigenvalues of the linearization of the  $\overline{BR}$  dynamics are of the form  $-1 \pm i\alpha/\gamma$ , where  $\alpha$  is a constant, possibly zero. Hence, the equilibrium is asymptotically stable. ■

We have seen that when the level of disturbance  $\gamma$  is very close to zero,  $\overline{BR}$  dynamics give the same results as  $PDA$  dynamics except in the zero sum case. It is natural to conjecture that, when the level of perturbation reaches zero, and we have the  $BR$  dynamics, the same result applies. This is indeed the case as was shown by Hofbauer (1995). However, we can now better understand why there is such a strong relationship between  $PDA$  and  $BR$  dynamics.<sup>4</sup> I give part of Hofbauer's results and I sketch his proof. Clearly as the  $BR$  dynamic is not differentiable at  $q$ , a linearization cannot be

<sup>4</sup>Gaunersdorfer and Hofbauer (1995) also show a close relationship between  $BR$  dynamics and the evolutionary replicator dynamics (which is  $PDA$ ) even when they both diverge from Nash equilibrium.

constructed. Rather Hofbauer designs an appropriate Liapunov function. At any time, the  $BR$  dynamic moves toward the best response, which, if strategy  $i$  currently has the highest payoff, will be the vertex of the simplex where  $x_i = 1$ . However, as we noted at the end of the previous section, if a game matrix is positive definite then an increase in  $x_i$  will increase  $(Ax)_i$ . Thus, the system can move a long way away from equilibrium before  $i$  ceases to be the best response. A converse argument holds when  $A$  is negative definite.

**PROPOSITION 2.** *If a game matrix  $A$  is negative definite with respect to  $\mathbb{R}_0^n$  or constant sum, any fully mixed equilibrium  $q$  is asymptotically stable under  $BR$  dynamics. If  $A$  is positive definite then  $q$  is unstable.*

*Proof.* This is a sketch of the proof due to Hofbauer (1995) for the special case where  $BR(x)$ , for any  $x \neq q$ , is one of the vertices of  $S_n$ . We label the vertices  $e_i$  for  $i = 1, \dots, n$ . Normalize  $e_i \cdot Ae_i = 0$  for all  $i$ . If  $j = \arg \max_i (Ax)_i$ , then use  $V(x) = \max_i (Ax)_i = e_j \cdot Ax$  as a Liapunov function. This will have a minimum at  $q$ . Given our normalization,  $\dot{V} = -V$ . If  $A$  is negative definite or zero sum,  $V(x) > 0$  for  $x \neq q$  and thus  $\dot{V} = -V < 0$  and  $q$  is asymptotically stable. However, if  $A$  is positive definite, given our normalization,  $q \cdot Aq < 0$  and thus close to  $q$ ,  $\dot{V} = -V > 0$ . ■

To understand these results better, consider the following game,

$$A = \begin{array}{|c|c|c|} \hline 0 & a & -b \\ \hline -b & 0 & a \\ \hline a & -b & 0 \\ \hline \end{array} \quad a, b > 0. \tag{10}$$

This version of the “Rock-Scissors-Paper” game has a unique mixed equilibrium at  $x = (1/3, 1/3, 1/3)$ . It can be shown that if  $a > b$  then  $A$  is negative definite with respect to  $\mathbb{R}_0^n$  and that if  $a < b$  then it is positive definite. While if  $a = b$ , the game is zero sum. Now we can see that in the first case, the mixed equilibrium is stable for all three dynamics  $PDA$ ,  $BR$ , and  $\overline{BR}$ . In the second case it is unstable for the  $PDA$ ,  $BR$  and, if the level of noise  $\gamma$  is sufficiently low,  $\overline{BR}$  dynamics. In the zero sum case, the  $BR$  and  $\overline{BR}$  dynamics will converge to the equilibrium. But for  $PDA$  dynamics the equilibrium is nonhyperbolic, or more specifically all eigenvalues of the linearization have zero real part, and it becomes impossible to obtain a general result on the behavior of the whole class of  $PDA$  dynamics. However, it is well known that for the evolutionary replicator dynamics, see, for example, Hofbauer and Sigmund (1988, p. 130), that the mixed equilibrium is neutrally stable.

There are similar considerations for asymmetric games. We can extend  $PDA$  dynamics to  $S_n \times S_m$  simply by writing

$$\dot{x} = Q(x)Ay, \quad \dot{y} = Q(y)Bx,$$

where  $Q(x)$  and  $Q(y)$  satisfy the conditions outlined in the previous section. We can extend the  $\overline{BR}$  dynamics in a similar manner with,

$$\dot{x} = \overline{BR}(y) - x, \quad \dot{y} = \overline{BR}(x) - y,$$

then the Jacobian taken at a perturbed equilibrium  $q'$  will be

$$\begin{pmatrix} 0 & \frac{d\overline{BR}(y)}{dy} \\ \frac{d\overline{BR}(x)}{dx} & 0 \end{pmatrix} - I = \frac{1}{\gamma} \begin{pmatrix} Q_B(x) & 0 \\ 0 & Q_B(y) \end{pmatrix} \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} - I,$$

where  $Q_B(x)$  and  $Q_B(y)$  are both *PDA* dynamic operators at  $q'$ . Again as in the symmetric case, if  $\mu$  is an eigenvalue for a *PDA* dynamic when linearized around a mixed equilibrium  $q$ , then we can find a  $\overline{BR}$  dynamic linearized at  $q'$  with an eigenvalue  $\mu/\gamma - 1$ . A simple application is the "Matching Pennies" type game considered by Fudenberg and Kreps (1993) among many others. Matching pennies is a  $2 \times 2$  zero sum game with a unique (mixed) equilibrium. Just as for the zero sum version of Rock-Scissors-Paper game (i.e., when  $a = b$ , see above), it is a center for the *PDA* dynamics (the linearization has purely imaginary eigenvalues). It follows that the equilibrium will be asymptotically stable under  $\overline{BR}$  dynamics. The point is that, in both symmetric and asymmetric games, the effect of noise is to push the path of the system inward from the boundary of the state space. Under this influence, the closed orbits surrounding a neutrally stable equilibrium point start to spiral inward and the equilibrium becomes asymptotically stable.

We can also relate the results here with those recently produced by Cressman (1997) on monotonic dynamics. The principal condition that a dynamic must satisfy to be called monotonic is that, in our present notation,  $\dot{x}_i/x_i > \dot{x}_j/x_j$  iff  $(Ax)_i > (Ax)_j$ . That is, the proportional growth rate of strategy  $i$  is greater than that of strategy  $j$  if and only if its current payoff is higher. Now, neither  $BR$  nor  $\overline{BR}$  dynamics meet this condition. This is because for best response dynamics the growth rate of a strategy  $\dot{x}_i$  is independent of its population share  $x_i$ . Nonetheless, we can make a connection. Cressman shows that the linearization of any smooth (i.e., differentiable) monotonic dynamic is a positive transformation of any hyperbolic linearization of the replicator dynamics. Or in other words the linearization of a monotonic dynamic at an interior mixed strategy equilibrium could be written  $QA$ , where  $Q$  is a *PDA* dynamic operator. Thus, *PDA* dynamics seem to be the linear form of both monotonic and best response dynamics.

## 5. CONCLUSION

There are two potential ways to conclude this article. One is to emphasize the differences between different formulations of evolutionary and learning dynamics and one is to emphasize their similarities. For example, if we concentrate our attention on the Rock-Scissors-Paper game (10) when  $a = b$ , or on asymmetric  $2 \times 2$  games with a unique mixed equilibrium, then we have the following conflicting results. The mixed equilibrium is asymptotically stable for the  $\overline{BR}$  and  $BR$  dynamics, and indeed for classical fictitious play. It may be asymptotically stable for some  $PDA$  dynamics, and unstable for others, it is certainly neutrally stable for some. Alternatively one can focus one's attention on the case where for (10)  $a > b$  and there is complete unanimity on the stability of the unique equilibrium. It does not matter whether one uses  $BR$ ,  $\overline{BR}$ , or  $PDA$  dynamics.

But even in the former case, it is not so easy to conclude that  $\overline{BR}$  and  $PDA$  dynamics are clearly differentiated. Particularly, I would resist the argument that best response dynamics are somehow superior because they converge in some cases where evolutionary type dynamics do not. In a recent article, Erev and Roth (1997) examine  $2 \times 2$  games with a unique mixed strategy equilibrium under reinforcement learning, a type of learning much more naive than fictitious play. Their simulations sometimes show convergence to equilibrium. This is despite earlier theoretical work which has shown that the expected motion of such a stochastic learning model is the same as the replicator dynamic and that learning does not converge to the mixed strategy equilibrium in  $2 \times 2$  games (Posch, 1997; Börgers and Sarin, 1997). Erev and Roth have, however, modified the basic reinforcement learning model by adding the idea that players experiment. This has the effect that the expected motion of the stochastic process is given by the replicator dynamics plus an additional stabilizing component. In fact, it is very similar in form to the linearization of  $\overline{BR}$  dynamic that we have constructed in this article. However, the exact connection will have to be established by further research.

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