Rank-Based Methods for the Analysis of Auctions*

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Abstract

A new method is proposed for the analysis of first price and all pay auctions, where bidding functions are written not as functions of values but as functions of the rank or quantile of the bidder's value in the distribution from which it was drawn. This method gives new results in both symmetric and asymmetric cases with independent values. It is shown that under this new method if one bidder has a stochastically higher distribution of values then her bidding function in terms of rank will always be higher than her rival's. This is a clearer result under weaker conditions than using standard methods. We also look at auctions where one bidder has more precise information than the other.

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1 Introduction

The importance of auctions in modern economic theory is enormous. In this paper, we suggest a different methodology for the analysis of auctions of the first price and all pay auctions. This method gives provides a new way of looking at auctions in both the symmetric and asymmetric cases. The standard approach to auctions is drawn from the theory of Bayesian games where strategies are defined as mappings from types to actions. In the case of auctions, this means that bidders have strategies which map values or signals of values to bids. Following Hopkins and Kornienko (2007b), we propose that instead that bidding functions are written not as functions of values but as functions of the rank or quantile of the bidder's value in the distribution from which it was drawn.[1]

We show that this new method can be simpler than the old, and thus allows us to derive clearer and less ambiguous results for both first price and all pay formats. Specifically, in an asymmetric setting, if one bidder has a stochastically higher distribution of values then her rival, then her bidding function in terms of rank will always be higher than her rival's and this is equivalent to the distribution of her bids being stochastically higher. In a symmetric setting, we have the comparative static result that, given a stochastically higher distribution of values, this will lead to a stochastically higher distribution of bids. Finally, in asymmetric auctions, it is usually assumed that one bidder has an advantage, because he has a higher average value for the object for sale. However, we find that rank based methods can also be used to look at the case where one bidder has more precise information about her rival's possible valuation than her rival has about her's.

To be clear, suppose bidders have values drawn from some distribution $G(v)$, then a bidder with value $\hat{v}$ has rank (also known as the quantile) $\hat{r} = G(\hat{v})$. If, as is normally assumed, G is strictly increasing, then there is a one-to-one relation between rank and value. So, it seems equally valid to consider this bidder's type as being $\hat{r}$ as much as it would be $\hat{v}$.[2]

The advantage of this approach is threefold.

1. It deals easily with different supports. Suppose we want to compare the bidding behaviour of two bidders $\{s, w\}$ with values drawn from two different distributions that have supports on [0,2] and [0,1] respectively. Does s bid more aggressively than w? It is difficult to say using traditional methods as w's bidding function $\beta_w(v)$ is defined only on [0,1] and therefore cannot be compared with $\beta_s(v)$ on

[1] This method has previously been used in first price auctions as an intermediate step in proving existence and uniqueness of equilibrium - see Lebrun (1999, p130). Recently, independently, Kirkegaard (2006) proposed an alternative method for the analysis of first price and all pay auctions based on the probability of winning rather than rank.

[2] In fact, the standard numerical method to generate a random draw from a distribution $G(v)$ is first to make a random draw r from [0,1] and then to find the value by setting $v = G^{-1}(r)$. That is, rank is decided first and value only second.

(1,2]. In contrast, using the rank-based methodology, we have bidding functions $b_s(r)$ and $b_w(r)$ that are both defined on [0,1] alone and so are directly comparable.

2. It requires weaker conditions. We show that it possible to order the bidding behaviour of different bidders using weaker assumptions than under classical methods. That is, simple stochastic dominance is sufficient rather than refinements such as reverse hazard rate dominance which have been necessary up to now.

3. It gives clearer results. In several cases, standard methods give ambiguous results. For example, in first price auctions where there is stochastic dominance but not reverse hazard rate dominance bidding functions can cross. In asymmetric all pay auctions, bid functions always cross even if one bidder is much stronger than the other. In contrast, we show that, even in these situations, rank based methods easily order different bidders' behaviour.

I should, however, make it clear anything proven using standard techniques will necessarily also hold when working with the rank-based approach. This is because working in terms of rank does not change the underlying game, the set of players or bidders, their strategy sets or payoffs. My argument is that rank indexing often allows existing results to be seen more clearly and because of that, it also allows some new results to be obtained very directly. In particular, while this paper's results on first price auctions are implicit in the analysis of Lebrun (1998, 1999) and Maskin and Riley (2000a), the results on all pay auctions and on the effect of more precise information are new.

2 Symmetric Auctions

In this section we will show rank based methods can aid the analysis of symmetric auctions, first price or all pay, with independent values, with or without risk aversion. We consider what happens when the distribution of values changes and find that a stochastically higher distribution of values will lead to uniformly higher bidding in terms of rank and a stochastically higher distribution of bids. This may seem unsurprising. However, this has not been shown using existing methods, and indeed we give examples where in the same circumstances writing bidding functions in terms of values gives ambiguous results.

In what follows, we consider auctions with n bidders. Each player has a valuation v drawn independently from a common distribution $G(v)$, which is twice differentiable with strictly positive density $g(v)$ on its support $[\underline{v}, \bar{v}]$. An agent of valuation v has utility $u(v-b)$ is she wins with a bid of b. She has utility $u(-Ib)$ if she loses, with I an indicator function being 0 in first price auctions and 1 in all pay auctions. We assume that u is strictly increasing and twice differentiable with $u'' \leq 0$. We use the standard normalisation $u(0) = 0$.

The innovation is that we will consider bidding strategies as a function of each player's rank not her valuation. The rank r of a bidder is equal to the rank of her valuation in the distribution of valuations, or $r = G(v)$. Equally, we can define $V(r) = G^{-1}(r)$, which gives a bidder's valuation as a function of her rank. Conventionally a bidding function would be $\beta(v)$, a mapping from valuation to bids. Here, we will consider equilibrium bidding strategies of the form $b(r)$. The important point is that as rank by definition runs from 0 to 1, the domain of the bidding function is [0, 1], that is $b(r) : [0,1] \to \mathbb{R}$, whatever the distribution of values.

One other important consideration is the following. There is a direct link between the bidding function defined in terms of rank $b(r)$ and the resulting distribution of bids $F(b)$.

Lemma 1 *The distribution of bids $F(b)$ generated by a strictly increasing bidding function in terms of rank $b(r)$ is equal to the inverse of the bidding function. That is, $F(b) = b^{-1}(b)$ or $b(r) = F^{-1}(r)$.*

Proof: Let B be the random variable representing a random draw from the distribution of bids generated by the bidding function $b(r)$. Then, $F(b) = \Pr[B \leq b] = \Pr[r \leq b^{-1}(b)] = b^{-1}(b)$. ∎

This leads to an important corollary. A bidding function generates a stochastically higher distribution of bids if and only if the bidding functions in terms of rank are ordered.

Corollary 1 *Let $F_i(b)$ be the distribution of bids generated by bidding function $b_i(r)$ for $i = s, w$. Then $F_s(b) \leq F_w(b)$ everywhere if and only if $b_s(r) \geq b_w(r)$ on [0,1].*

2.1 First Price

We consider a standard n bidder first price auction. We look for a symmetric equilibrium in which all bidders use the same strategy $b(r)$. Suppose a bidder bids according to the proposed equilibrium strategy then another bidder of rank r who bids as though she had rank $\hat{r}$ or $b(\hat{r})$ would win with probability $\Pr[b(\hat{r}) > b(r)] = \Pr[\hat{r} > b^{-1}(b(r))] = \Pr[\hat{r} > r] = \hat{r}$. Facing $n-1$ other bidders who bid using the strategy $b(r)$, that bidder would then have expected utility

$$\hat{r}^{n-1}u(V(r) - b(\hat{r})) \tag{1}$$

For the bidder not to want to bid $b(\hat{r})$ in place of $b(r)$, the following first order condition must hold

$$(n-1)\hat{r}^{n-2}u(V(r) - b(\hat{r})) - b'(\hat{r})\left(\hat{r}^{n-1}u'(V(r) - b(\hat{r}))\right) = 0 \tag{2}$$

For a symmetric equilibrium, we set $\hat{r}$ equal to r. Rearranging this gives us the following differential equation, the solution to which constitutes a symmetric equilibrium:

$$b'(r) = \frac{(n-1)}{r}\frac{u(V(r)-b)}{u'(V(r)-b)} = \frac{(n-1)}{r}\psi(V(r),b) \tag{3}$$

with boundary condition $b(0) = V(0)$.[3] In the case of risk neutrality, this differential equation has the explicit solution

$$b(r) = V(r) - \frac{\int_0^r t^{n-1}\, dV(t)}{r^{n-1}} \tag{4}$$

How does this compare to the standard approach? Let $\beta(v)$ be the equilibrium bidding function in terms of values. Note that the two bidding functions are linked in the following way: $\beta(V(r)) = b(r)$ or equivalently $\beta(v) = b(G(v))$. That is, to convert from values to rank, replace v with $V(r)$; to convert from rank to values, replace r with $G(v)$. Furthermore, we can regain the standard differential equation in terms of valuations through the relation $\beta'(v) = d\beta/dv = db/dr \cdot dr/dv = b'(r)g(v)$. That is,

$$\beta'(v) = (n-1)\frac{g(v)}{G(v)}\frac{u(v-\beta)}{u'(v-\beta)} = (n-1)\frac{g(v)}{G(v)}\psi(v,\beta) \tag{5}$$

with boundary condition $\beta(\underline{v}) = \underline{v}$.

The important point is that if we compare (3) and (5) is that the density $g(v)$ is absent in (3) and indeed the distribution of values only enters through the function $V(r) = G^{-1}(r)$. In contrast, in the traditional approach the bidding function will depend on $g(v)/G(v)$ which is known as the reverse hazard ratio. Thus, in order to rank bids and to do comparative statics exercises typically one needs to use the reverse hazard rate order which is a stronger condition than stochastic dominance (see, for example, Lebrun (1998), Krishna (2002); Maskin and Riley (2000a) call a similar property "conditional stochastic dominance").

Working in terms of rank makes the analysis of changes in the distribution of valuations on bidding behaviour much easier. In particular, the reverse hazard rate order is not needed, rather simple stochastic dominance is sufficient to order bidding functions and revenue.

Proposition 1 *Let $G_A(v)$ and $G_B(v)$ be two distributions of values with differentiable inverse functions $V_i(r) = G_i^{-1}(r)$ for $i = A, B$ such that $V_A(r) > V_B(r)$ for all $r \in (0,1)$ (which implies G_A stochastically dominates G_B). There is a unique equilibrium bidding function $b_i(r)$ that solves the differential equation (3) under distribution G_i for $i = A, B$. Furthermore, $b_A(r) > b_B(r)$ for all $r \in (0,1)$ and $F_A(b)$ stochastically dominates $F_B(b)$. Expected revenue is strictly higher under distribution A.*

[3]Strictly speaking, the boundary condition is that $\lim_{r\to 0} b(r) = V(0)$ with the bidder with the lowest valuation being indifferent over all bids in the range $[0, V(0)]$ as in symmetric equilibrium he never wins. However, it simplifies the analysis to concentrate on the case where in fact $b(0) = V(0)$ without any great loss in generality.

Proof: Existence and uniqueness carry over from the standard results (see, for example, Maskin and Riley (2000b)) under value indexing as whether you consider values or ranks, it is the same game and has the same equilibria. Then, note that as $\psi(v,b) = u(v-b)/u'(v-b)$,

$$\frac{d\psi(v,b)}{dv} = \frac{(u'(\cdot))^2 - u''(\cdot)u(\cdot)}{(u'(\cdot))^2} > 0.$$

Hence, if there is any point of crossing of $b_A(r)$ and $b_B(r)$ at some $\tilde{r} \in (0,1)$, as $V_A(\tilde{r}) > V_B(\tilde{r})$ then $b'_A(\tilde{r}) > b'_B(\tilde{r})$. Therefore, $b_A(r)$ can only cross $b_B(r)$ from below and there can be only one such crossing. Now, if $V_A(0) > V_B(0)$, the lowest value under distribution A is higher than the lowest under B, then by the boundary condition $b_A(0) > b_B(0)$ and we are done. If $V_A(0) = V_B(0)$ ($V_A(0) < V_B(0)$ is not consistent with our assumption that distribution A is stochastically higher than B) and thus $b_A(0) = b_B(0)$, given b_A and b_B can cross only once, then our result can only fail if $b_B(r) \geq b_A(r)$ on the interval $[0, \tilde{r}]$ for some $\tilde{r} > 0$. But as $d\psi/db = -d\psi/dv < 0$, and as $V_A(r) > V_B(r)$ and $b_B(r) \geq b_A(r)$ on $(0, \tilde{r})$, we must have $b'_A(r) > b'_B(r)$ on that interval. Given $b_A(0) = b_B(0)$, this would imply that $b_A(r) > b_B(r)$ on $(0, \tilde{r})$ which is a contradiction. We have shown that $b_A(r) > b_B(r)$ on $(0,1)$. It then follows that $F_A(b)$ stochastically dominates $F_B(b)$ from Lemma 1. Expected revenue is equal to the expectation of the n-th order statistic of the distribution of bids. Stochastic dominance implies that the order statistics of distributions $F_A(b)$ and $F_B(b)$ are also stochastically ordered (see, for example, Krishna (2002, p266)) and the result follows. ∎

This result does not hold under the traditional case of looking at bidding functions in terms of values as the next example shows. The point is that stochastic dominance, as opposed to reverse hazard rate dominance, is not sufficient to order bidding functions in terms of values. Here we give an example where there is stochastic dominance but the higher distribution gives a lower bidding function in terms of values. Of course, by Proposition 1, by moving to ranks, we can resolve this ambiguity.

Example 1 *Let $G_A(v) = 3v - 2v^2$ which has support on [0, 0.5] and $G_B(v) = 3v - v^2$ with support on [0, 0.382] (this is drawn from Maskin and Riley (2000a)). Then $G_A(v) \leq G_B(v)$ and so G_A stochastically dominates G_B. However, G_A does not reverse hazard rate dominate G_B as $G_A(v)/G_B(v)$ is actually decreasing on [0,0.382]. For the risk neutral case, it can be calculated that $\beta_A(v) < \beta_B(v)$ on [0, 0.382] where β_i is the bidding function corresponding to distribution G_i. That is, the higher distribution generates lower bids for a given valuation. Of course, we can't compare the two bidding functions on the interval (0.382,0.5] as $\beta_B(v)$ is not defined there. And it is not clear which distribution generates higher revenue.*

But if we look at the same problem in terms of rank, things are much clearer. First, by Proposition 1, $b_A(r) > b_B(r)$ for all $r \in (0,1)$, the stochastically higher distribution gives higher bids for a given rank. Second, equally by Proposition 1, the distribution of bids is stochastically higher under G_A than under G_B and so revenue is definitely higher.

2.2 All Pay

We now consider all pay auctions and find that the contrast between rank-indexing and traditional methods is even greater. As with first price auctions, ordering bidding functions in terms of rank is straightforward. However, no matter what stochastic order is used, ordering bidding functions in terms of value is not possible.

Again if all bidders but one bid according to the smooth increasing bidding function $b(r)$ and the other bids $b(\hat{r})$, that bidder expects utility equal to

$$\hat{r}^{n-1}u(V(r) - b(\hat{r})) + (1 - \hat{r}^{n-1})u(-b(\hat{r})) \tag{6}$$

Differentiating with respect to $\hat{r}$ gives us the first order condition

$$(n-1)\hat{r}^{n-2}\left(u(V(r) - b) - u(-b)\right) - b'(\hat{r})\left(\hat{r}^{n-1}u'(V(r) - b) + (1 - \hat{r}^{n-1})u'(-b)\right) = 0 \tag{7}$$

Setting $\hat{r}$ equal to r and rearranging gives us the differential equation

$$b'(r) = \frac{(n-1)r^{n-2}\left(u(V(r) - b) - u(-b)\right)}{r^{n-1}u'(V(r) - b) + (1 - r^{n-1})u'(-b)} = (n-1)r^{n-2}\gamma(r, V(r), b) \tag{8}$$

with the boundary condition in this case $b(0) = 0$. Again, in the case of risk neutrality, there is an explicit solution

$$b(r) = V(r)r^{n-1} - \int_0^r t^{n-1}\, dV(t) \tag{9}$$

We can again compare this with the corresponding differential equation in terms of values:

$$\beta'(v) = \frac{(n-1)g(v)G(v)^{n-2}\left(u(v - \beta) - u(-\beta)\right)}{G(v)^{n-1}u'(v - \beta) + (1 - G(v)^{n-1})u'(-\beta)} = (n-1)g(v)G(v)^{n-2}\gamma(G(v), v, \beta) \tag{10}$$

We show again that using rank-based methods, simple stochastic dominance is sufficient to order bidding behaviour.

Proposition 2 *Let $G_A(v)$ and $G_B(v)$ be two distributions of values with differentiable inverse functions $V_i(r) = G_i^{-1}(r)$ for $i = A, B$ such that $V_A(r) > V_B(r)$ for all $r \in (0,1)$ (which implies G_A stochastically dominates G_B). Then, there is a unique equilibrium bidding function $b_i(r)$ that solves the differential equation (8) under distribution G_i for $i = A, B$. Furthermore, $b_A(r) > b_B(r)$ for all $r \in (0,1)$ and $F_A(b)$ stochastically dominates $F_B(b)$. Expected revenue is strictly higher under distribution A.*

Proof: Existence and uniqueness are standard (see Amann and Leininger (1996)). It is not difficult to establish that, for the function $\gamma(r, v, b)$ as given in (8), it holds that $\partial\gamma(r,v,b)/\partial v > 0$ and that $\partial\gamma(r,v,b)/\partial b < 0$. The proof then follows that of Proposition 1. ∎

Note that in all-pay auctions, the contrast in behaviour of bidding functions in terms of values and rank is much greater. In particular, even using the strongest stochastic order, it is impossible to rank bidding functions in terms of values. This is because, in contrast with bidding functions in terms of rank, the bidding functions in terms of values cross, as we see in this example.

Example 2 *Suppose $G_A(v) = v^2$ and $G_B(v) = v$ then G_A stochastically dominates and reverse hazard rate dominates G_B. However, if there are two risk neutral bidders the bidding functions for the all pay auction in terms of values are $\beta_A(v) = 2v^3/3$ and $\beta_B(v) = v^2/2$ which cross at $v = 3/4$. The stochastically higher distribution induces lower bids for a given value at low values, but higher bids at high values. But by Proposition 2, it must be different in terms of rank. Indeed, the bidding functions in terms of rank are $b_A(r) = 2r^{3/2}/3$ and $b_B(r) = r^2/2$. Thus, the bidding function corresponding to the stochastically higher distribution is always higher, that is, $b_A(r) > b_B(r)$ on $(0,1]$.*

We can show that the crossing behaviour in the example for bidding in terms of values generalises. We impose the strongest known refinement of stochastic dominance, likelihood ratio dominance, which implies both reversed hazard ratio dominance and stochastic dominance (again, see Krishna (2002)). Yet, the bidding functions in terms of values cross.

Proposition 3 *Suppose there are two distributions of values $G_A(v)$ and $G_B(v)$ with the same support $[\underline{v},\bar{v}]$ such that $G_A(v)$ likelihood ratio dominates $G_B(v)$. Then the corresponding equilibrium bidding functions in terms of values for the all pay auction $\beta_A(v)$ and $\beta_B(v)$ cross at least once on $(\underline{v},\bar{v})$ with $\beta_B(v) > \beta_A(v)$ on the interval $(\underline{v},\hat{v})$ where $\hat{v}$ is the unique point in $(\underline{v},\bar{v})$ such that $g_A(v) = g_B(v)$.*

Proof: This is Corollary 4 in Hopkins and Kornienko (2007a). ∎

This result is illustrative. There are two simple extensions which we do not give here. For simplicity we have assumed a common support, but similar results hold for differing supports. Furthermore, in the case of risk neutrality, it is possible to show that there is exactly one crossing with $\beta_A(v) > \beta_B(v)$ for high values.

3 Asymmetric Auctions

In this section, we consider a two bidder first price auction, where one bidder is "stronger" than the other. We show that again rank-based methods can be used to rank bidding functions under weaker conditions than in traditional analysis.

There are two bidders, strong and weak $\{s, w\}$. In this section, we assume that they are risk neutral. The strong bidder has values distributed according to $G_s(v)$ on $[\underline{v}_s, \bar{v}_s]$ and the weak according to $G_w(v)$ on $[\underline{v}_w, \bar{v}_w]$ with $\underline{v}_s \geq \underline{v}_w$ and $\bar{v}_s \geq \bar{v}_w$. Let $F_s(b)$ give the distribution of bids by the strong bidder, $F_w(b)$ by the weak. In equilibrium, they have the same maximum bid $\bar{b}$.[4]

Note that if the weak bidder bids according to a strictly increasing function $b_w(r)$ then the probability that the strong bidder wins with a bid b_s would be

$$\Pr[b_s > b_w(r)] = \Pr[b_w^{-1}(b_s) > r] = b_w^{-1}(b_s)$$

This relationship will make it more convenient to work with inverse bid functions. This is standard in the literature. However, the difference here is that, given Lemma 1, the inverse bidding function is equal to the distribution function of bids. That is, $b_w^{-1}(b_s) = F_w(b_s)$.

3.1 First Price

First, consider the first price auction. If the strong bidder bids b_s when the weak bidder bids according to the strategy $b_w(r)$, the strong bidder has expected utility

$$b_w^{-1}(b_s)(V_s(r) - b_s) = F_w(b_s)(V_s(r) - b_s) \tag{11}$$

Differentiating with respect to b_s, one obtains a first order condition

$$F_w'(b_s)(V_s(r) - b_s)) - F_w(b_s) = 0$$

If we look at the weak bidder's problem, we can obtain a similar differential equation from her first order condition. Putting them together we have these simultaneous differential equations in b

$$F_s'(b) = \frac{F_s(b)}{V_w(F_w(b)) - b}\,;\; F_w'(b) = \frac{F_w(b)}{V_s(F_s(b)) - b} \tag{12}$$

The boundary conditions if $\underline{v}_w = \underline{v}_s$ are $F_s(\underline{b}) = F_w(\underline{b}) = 0$ with $\underline{b} = \underline{v}_w$. If $\underline{v}_w < \underline{v}_s$, then $F_w(\underline{b}) = G_w(\underline{b})$ and $\underline{b} = \max \arg\max_b(\underline{v}_s - b)F_w(b)$. That is, in the second case, the support of winning bids is $[\underline{b}, \bar{b}]$. The weaker bidder bids her value when she has a value in $[\underline{v}_w, \underline{b}]$ but never wins. The value of $\underline{b}$ is set by the strong bidder's best response conditional on having the value $\underline{v}_s$ and given the weak bidder's behaviour (see Maskin and Riley (2000a)).

We can show that if the strong bidder has a stochastically higher distribution of values then her bidding function in terms of rank is always higher than that of the weak bidder. This implies that her distribution of bids is stochastically higher. Note that the

[4] See Maskin and Riley (2000a).

result that stochastically higher distribution of values implies a stochastically higher distribution of bids has already been established by Lebrun (1998, 1999) and Maskin and Riley (2000a). However, to order bidding functions in terms of values, this previous work needs to use the stronger assumption of reverse hazard rate dominance.

Proposition 4 *Suppose $V_s(r) > V_w(r)$ on $(0,1)$, equivalently $G_s(v) < G_w(v)$ on $(\underline{v}_w, \bar{v}_s)$ and so G_s stochastically dominates G_w. Then $b_s(r) > b_w(r)$ on (0,1) and $F_s(b) < F_w(b)$ on $(\underline{b}, \bar{b})$ so that $F_s(b)$ stochastically dominates $F_w(b)$.*

Proof: Since it is easier to work with the inverse bidding functions, we show that $F_s(b) < F_w(b)$ on $(\underline{b}, \bar{b})$. Suppose there is a point $\hat{r} \in (0,1)$ such that the two bidding functions cross, that is, $b_s(\hat{r}) = b_w(\hat{r}) = \hat{b}$. This implies also that $F_s(\hat{b}) = F_w(\hat{b}) = \hat{r}$. At such a point, from (12) we have

$$F_s'(\hat{b}) = \frac{\hat{r}}{V_w(\hat{r}) - \hat{b}} > \frac{\hat{r}}{V_s(\hat{r}) - \hat{b}} = F_w'(\hat{b}) \tag{13}$$

That is, at any crossing point of the inverse bidding functions (equivalently the distribution functions of bids) $F_s(b)$ is steeper. Hence, there can only be one such crossing. Thus, our claim fails if there is such a crossing or if $F_s(b) \geq F_w(b)$ everywhere. In either case, it must be that $F_s(b) > F_w(b)$ on some interval $(\tilde{b}, \bar{b})$ for some $\tilde{b} \geq \underline{b}$ (given $F_s' > F_w'$ at any point of crossing, we can rule out equality between F_s and F_w). So it must be that $V_s(F_s(b)) > V_w(F_w(b))$ on that interval. It follows from (12) that we have $F_s'(b) > F_w'(b)$ on $[\tilde{b}, \bar{b}]$. Given $F_s(\tilde{b}) = F_w(\tilde{b})$, this would imply $F_s(\bar{b}) > F_w(\bar{b})$, which is not possible. ∎

Note that this apparently completely reverses the results obtained using value-based methods. Here, we have the stronger bidder bidding more than the weaker bidder at each rank. However, under the traditional approach and under the stronger assumption of the reverse hazard rate order, the weaker bidder bids more for a given value. Let us see a concrete example of this.

Example 3 *This example is from Krishna (2002, pp49-51). The strong bidder's value is distributed uniformly on [0, 4/3], the weak's uniformly on [0, 4/5]. The equilibrium bidding functions in terms of values are $\beta_s(v) = (-1 + \sqrt{1+v^2})/v$ and $\beta_w(v) = (1 - \sqrt{1-v^2})/v$ and $\beta_w(v) > \beta_s(v)$ on the intersection of their support, that is (0, 4/5]. That is, in terms of values, the weaker bidder will bid more than the strong bidder.*

However, when we move to ranks, we see a different picture. We have $b_s(r) = (-3 + \sqrt{9+16r^2})/(4r)$ and $b_w(r) = (5 - \sqrt{25-16r^2})/(4r)$ and $b_s(r) > b_w(r)$ on (0, 1). That is, for a given rank in each's distribution, the strong bidder bids more. This is illustrated in Figure 1.

Note, however, that in both cases we are talking about the same, unique equilibrium. The use of rank-based methods is way of looking at existing models in a new light.

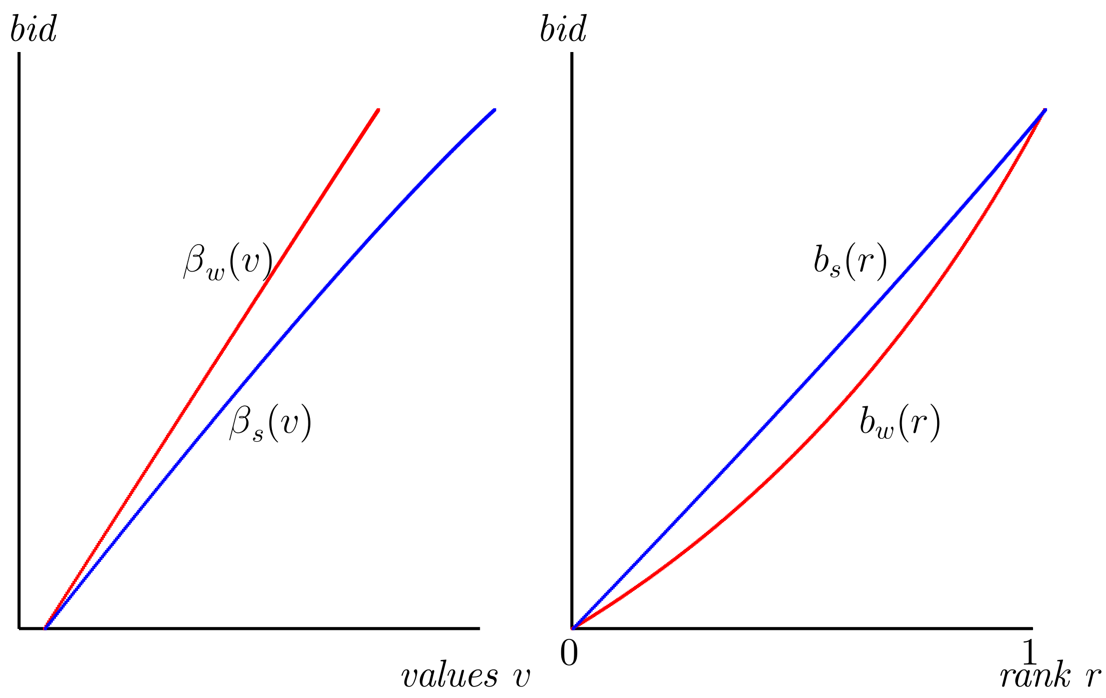

Figure 1: Which bidder is the more aggressive? In terms of values, the bidding function of the strong bidder $\beta_s(v)$ is less than the bidding function of the weak bidder $\beta_w(v)$. But this relation is reversed when we look at bidding in terms of ranks where $b_s(r) > b_w(r)$.

Finally, rank based methods again give clear answers where value-based results are ambiguous.

Example 4 *Let us take the two distributions from Example 1, so that* $G_s(v) = 3v - 2v^2$ *which has support on [0, 0.5] and* $G_w(v) = 3v - v^2$ *with support on [0, 0.382]. As Maskin and Riley (2000a) point out,* G_s *stochastically dominates* G_w*, but* G_s *does not reverse hazard rate dominate* G_w *and the corresponding bidding functions in terms of values* $\beta_s(v)$ *and* $\beta_w(v)$ *will cross. However, by Proposition 4, the bidding functions in terms of rank are clearly ordered so that* $b_s(r) > b_w(r)$ *on (0, 1).*

3.2 All Pay

In the all pay auction, if the strong bidder bids b_s when the weak bidder bids according to the strategy $b_w(r)$, the strong bidder has expected utility

$$b_w^{-1}(b_s)V_s(r) - b_s = F_w(b_s)V_s(r) - b_s \tag{14}$$

Differentiating with respect to b_s, one obtains a first order condition

$$F_w'(b_s)V_s(r) - 1 = 0$$

If we look at the weak bidder's problem, we can obtain a similar differential equation from her first order condition. Putting them together we have these simultaneous differential equations in b

$$F_s'(b) = \frac{1}{V_w(F_w(b))} \; ; \; F_w'(b) = \frac{1}{V_s(F_s(b))} \tag{15}$$

The boundary conditions in this case are $F_s(0) = 0, F_w(0) = k \geq 0$. That is, the weaker bidder may have a mass point at 0 and bid zero with positive probability (Amann and Leininger (1996)).

Proposition 5 *Suppose $V_s(r) > V_w(r)$ on $(0,1)$, equivalently $G_s(v) < G_w(v)$ on $(\underline{v}_w, \bar{v}_s)$ and so G_s stochastically dominates G_w. Then in the all pay auction there is a unique equilibrium where $b_s(r) > b_w(r)$ on (0,1) and $F_s(b) < F_w(b)$ on $(\underline{b}, \bar{b})$ so that $F_s(b)$ stochastically dominates $F_w(b)$.*

Proof: Existence and uniqueness is proven in Amann and Leininger (1996). From the differential equations (12) and the assumption that $V_s(r) > V_w(r)$ on $(0,1)$, it is clear that there can be only one crossing of $F_s(b)$ and $F_w(b)$ on $(0,1)$ and $F_s(b)$ must be steeper at such a point. The proof then follows that of Proposition 4. ∎

In contrast, one can show that, in a similar way to Proposition 3, the bidding functions in terms of ranks must cross in the all pay auction. We give an example.

Example 5 *This is from Amann and Leininger (1996). The strong distribution is $G_s(v) = v^5$ and the weak $G_w(v) = v^2$ both on [0,1]. It is easy to check that G_s likelihood ratio, reverse hazard rate and stochastically dominates G_w. The strong bidder bids $\beta_s(v) = (27v^5 + 25v^9)/72$ which is very flat and low until near 1. The weak bidder bids nothing for values on [0,0.375] but then her bidding function rises steeply and almost immediately crosses the strong bidder at $v = 0.389$ and then remains higher until $v = 1$. What this hides is that given $G_s(v) = v^5$, there is almost no probability mass on low values of v. That is, the low values of $\beta_s(v)$ for low values of v are almost irrelevant as the strong bidder almost never has a low valuation. Putting the bid functions in terms of rank, in effect weights the bidding functions in terms of probability and we know from Proposition 5 that $b_s(r)$ will always be higher than $b_w(r)$, even though the bidding functions in terms of values do cross.*

4 Better Information

Up to now, we have looked at asymmetric auctions where bidders are ordered in terms of valuations. We now look at a first price auction where one bidder has better information. The possibility of comparative statics using conditions weaker than (first order) stochastic dominance has not received much attention, exceptions being Hopkins and Kornienko (2007a) and Kirkegaard (2006).

Let us take the interpretation that $G_s(v)$ is the weak bidder's prior about the strong bidder's type and $G_w(v)$ is prior of the strong bidder about the weak. Then we assume that strong bidder has a prior that is less dispersed than the prior of weak bidder, so that $G_s(v)$ is more dispersed than $G_w(v)$. That is, the strong bidder has more precise information about the possible valuation of his rival. We find that in this case the strong bidder bids more than the weak for most ranks, with the strong bidder bidding more only at very low ranks. The intuition is that as as one's prior becomes more compressed, the gain in the probability of winning the auction by raising one's bid by a penny increases. As the possible types of one's opponent become more closely packed, the easier it is to surpass him, and thus one should bid more.

To show this, we will use a version of the dispersion order (see Shaked and Shanthikumar (1994)). Two continuously differentiable distributions G_s and G_w can be ordered in terms of dispersion in the following way

$$G_s \geq_d G_w \text{ if and only if } g_s(G_s^{-1}(r)) \leq g_w(G_w^{-1}(r)) \Leftrightarrow V_s'(r) \geq V_w'(r) \text{ for all } r \in (0,1) \tag{16}$$

That is, for a fixed rank, the more dispersed distribution is less dense than the less dispersed one. A simple example of this is two uniform distributions with one with strictly smaller support than the other. Note that because the condition is expressed in terms of ranks, there is no problem in comparing distributions with different, even disjoint, supports.

Here we combine the density condition (16) with a single crossing condition on the distribution functions. A simple example that fits this pattern would be G_s being uniform on $[0,1]$ and G_w being uniform on $[0.25, 0.75]$. We show that this implies that the bidding functions are also single crossing with the stronger bidding more at low rank.

Proposition 6 *Suppose $V_s'(r) > V_w'(r)$ on $[0,1]$ and that there is one point $\tilde{r} \in (0,1)$ such that $V_w(\tilde{r}) = V_s(\tilde{r})$. This implies that the distribution G_s is more dispersed than G_w in terms of the dispersion order. Then $b_s(r) < b_w(r)$ on $[0, r_0)$ and $b_s(r) > b_w(r)$ on $(r_0, 1)$ where $r_0 < \tilde{r}$.*

Proof: Again we use inverse bid functions (equivalently the distribution functions of bids). First, given that $V_s'(r) > V_w'(r)$ and V_s and V_w are single crossing, this implies $\underline{v}_s = V_s(0) < V_w(0) = \underline{v}_w$. Then, by the boundary conditions for the differential equation (12), we have $\underline{b} > \underline{v}_s$ and $F_s(\underline{b}) > F_w(\underline{b}) = 0$. As $V_s(r) < V_w(r)$ on $[0, \tilde{r})$, by the relation (13), at any crossing point of $r_0 = F_s(b) = F_w(b)$ where $r_0 \in (0, \tilde{r})$, it must be that $F_s' < F_w'$. Thus, there can be at most one crossing r_0 in $(0, \tilde{r})$.

Turning to possible points of crossing in $(\tilde{r}, 1)$, as $V_s(r) < V_w(r)$ on $(\tilde{r}, 1]$, at any such crossing point it must be that $F_s' > F_w'$. Hence if there is no crossing on $(0, tilder)$

then it must be that $F_s(b) > F_w(b)$ on $(\underline{b}, \bar{b})$ (given $F_s' > F_w'$ at any point of crossing, we can rule out equality between F_s and F_w). So it must be that $V_s(F_s(b)) > V_w(F_w(b))$ on an interval $[\tilde{b}, \bar{b}]$ where $\tilde{b}$ is chosen so that $F_w(\tilde{b}) > \tilde{r}$. It follows from (12) that we have $F_s'(b) > F_w'(b)$ on $[\tilde{b}, \bar{b}]$. Given $F_s(\tilde{b}) > F_w(\tilde{b})$, this would imply $F_s(\bar{b}) > F_w(\bar{b})$, which is not possible. So there must be a crossing on $(0, \tilde{r})$ and $b_s(\tilde{r}) > b_w(\tilde{r})$.

Finally, for $r > \tilde{r}$, $V_w(r) < V_s(r)$, so there can only be one crossing on $[\tilde{b}, \bar{b}]$ and $F_s(b)$ can only cross $F_w(b)$ from below and therefore it must happen only at $\bar{b}$. Thus, there is no crossing on $(\tilde{r}, 1)$. ∎

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